Chapter 11 Infinite Sequences and Series

Section 11.1 ***Sequences***

Def: A sequence can be thought as a list of numbers written in a definite order; $a_1, a_2, a_3, ..., a_n, ...$

Def: An infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer n.

Notation: $f(n) = \{f(1), f(2), ..., f(n), ...\} = \{a_1, a_2, a_3, ..., a_n, ...\} = \{a_n\}_{n=1}^{\infty}$

Ex: List the first 4 terms of the following sequences:

a) $\{a_n\} = \left\{\dfrac{2n+1}{n^2+3}\right\}$

b) $\{b_n\} = \left\{\dfrac{(-1)^n}{(n+1)!}\right\}$

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c) $\{c_n\} = \{\cos(n\pi)\}$

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d) $d_1 = 2;\ d_2 = -1;\ d_{n+2} = 2d_{n+1} + d_n - n!$

General formula of a sequence:

Ex: Put the following sequence into its general formula

a) $\sqrt{2}, \sqrt{3}, \sqrt{4}, ..., \sqrt{n}, ... =$

b) $\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n} .... =$

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., \frac{n-1}{n}, ... =$

d) $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ..., (-1)^{n+1} \frac{1}{n}, ... =$

e) $\left\{-\frac{2}{3},\frac{3}{9},-\frac{4}{27},\frac{5}{81},\ldots,\frac{(-1)^n(n+1)}{3^n}\right\}=$

f) $\left\{\frac{3}{5},-\frac{4}{25},\frac{5}{125},-\frac{6}{625},\frac{7}{3125},\ldots\right\}=$

Ex: Recursive Formula: (Fibonacci Sequence)

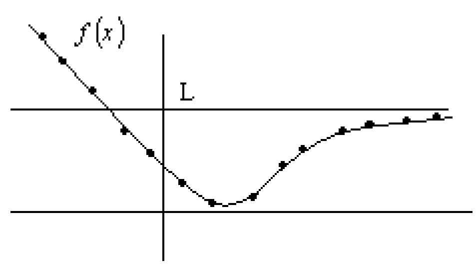
$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, ...\}$

Limit of a sequence: where does a sequence go to? i.e. what is the number (only one) that a sequence will be eventually approaches to.

Def: A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$. If we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent)

A more precise version of limit:

A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ whenever n>N.



Def: Diverges to Infinity:
The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N such that for all n larger than N , $a_n > n$. If this condition holds we write $\lim_{n\to\infty} a_n = \infty$

Theorem 1: $\lim_{n\to\infty} a_n = A;\ \lim_{n\to\infty} b_n = B$

$$a)\quad \lim_{n\to\infty}\left(a_n \pm b_n\right) = A \pm B;$$

$$b)\quad \lim_{n\to\infty}\left(a_n b_n\right) = AB;$$

$$c)\quad \lim_{n\to\infty}\left(ka_n\right) = kA;$$

$$d)\quad \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B};\ B \neq 0$$

Squeeze Theorem : $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Theorem 3: Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \to f(L)$

Theorem 4: Suppose that $f(x)$ is a function defined for all $x \ge x_0$ and that $\{a_n\}$ is a sequence of real numbers such that $f(n) = a_n$ for all $n \ge n_0$. Then
$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$

Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$

Ex: Evaluate the limit of the following sequences:

a) $\{a_n\} = \left\{ \dfrac{2n^3 - 2n + 7}{\sqrt{9n^6 - 5n + 3}} \right\}$

b) $\{b_n\} = \left\{ \left(1 + \dfrac{3}{n}\right)^{2n} \right\}$

c) $\{c_n\} = \left\{ \dfrac{\sin^2(2n+1)}{\sqrt{n^3 + 2}} \right\}$

d) $\{d_n\} = \{1 + (-1)^n\}$.

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e) $\{e_n\} = \{\sin(n)\}$

f) $\{a_n\} = \left\{\frac{9^n + 7^n}{9^n + 5^n}\right\}$

g) $\{a_n\} = \{\ln(n+2) - \ln(3n+2)\}$

h) $\{a_n\} = \left\{\dfrac{n!}{n^n}\right\}$

g) $\{a_n\} = \{\sqrt[n]{n}\}$

Note: For what values of r is the sequence $\{r^n\}$ convergence?

Sol: $\lim_{n\to\infty} r^n = \begin{cases} \infty \text{ if } r > 1 \\ 1 \text{ if r} = 1 \\ 0 \text{ if } 0 < \text{r} < 1 \end{cases}$ Demonstrate this by plotting point for n.

__Theorem__: The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if r} = 1 \\ 0 & \text{if -1} < \text{r} < 1 \end{cases}$$

__Monotone and Bounded Sequences__:

__Def__: Let $\{a_n\}$ be a sequence of real numbers.

- The sequence is monotone increasing if $a_n \le a_{n+1}$ for all $n \ge 1$
- The sequence is monotone decreasing if $a_n \ge a_{n+1}$ for all $n \ge 1$
- The sequence is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$
- The sequence is bounded below if there is a number m such that $a_n \ge M$ for all $n \ge 1$

(If a sequence is bounded above and bounded below, we say that the sequence is bounded. If a sequence is not bounded, we say that it is unbounded.)

__Ex:__ a) For positive integer n, let $a_n = \sqrt{n^4 + n^3} - n^2$. Show that the sequence $\{a_n\}$ is monotone increasing and unbounded.

b) Let $\{a_n\} = \left\{ \frac{(-1)^n}{n} \right\}$. Show that the sequence $\{a_n\}$ is bounded but not monotone.

c) Show that the sequence $a_n = \frac{3}{n+5}$ is monotone decreasing.

d) Show that the sequence $a_n = \frac{n}{n^2+1}$ is monotone decreasing.

c) $\{a_n\} = \left\{\frac{3}{n+5}\right\}$ ➔ it's monotone decreasing.

d) $\{a_n\} = \left\{\frac{n}{n^2+1}\right\}$

<u>The Monotone Sequence Theorem</u>: Let $\{a_n\}$ be a monotone increasing sequence of real numbers.

a) if $\{a_n\}$ is bounded above, then $\lim_{n\to\infty} a_n$ exists.

b) if $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$

Ex: Define a sequence $\{a_n\}$ by the recursion relationship $a_1 = 1$; $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that the sequence converges and find its limit.

Ex: Investigate the sequence $\{a_n\}$ defined by the recursive definition

$a_1 = 2; a_{n+1} = \frac{1}{2}(a_n + 6)$ for n = 1, 2, 3,…

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1$ (x>0)

4. $\lim_{n\to\infty} x^n = 0 \ (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

Ex: Evaluate the following limits:

a) $\lim_{n\to\infty} \frac{\ln(n^3)}{4n}$

b) $\lim_{n\to\infty} \sqrt[n]{n^2}$

c) $\lim_{n\to\infty} \left(\frac{3^n}{n^3}\right)$

d) $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} =$

e) $\lim_{n\to\infty}(n+4)^{1/(n+4)}$

f) $\lim_{n\to\infty}\frac{(10/11)^n}{(9/10)^n+(11/12)^n}$

g) $\lim_{n\to\infty}\frac{3^n 6^n}{2^{-n}n!}$;

h) $\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n$