Series

Def:

Partial Sum:

Def: Given a series $\sum_{n=1}^{\infty} a_n$, let ..$S_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{i=1}^{n} a_i$ be the n^th^ partial sum of the series. If the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n \to \infty} S_n = S$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$\sum_{n=1}^{\infty} a_n = S$, the number S is called the sum of the series. Otherwise, the series is called divergent.

Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$

Ex: Evaluate the following:

a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

b) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots=$

c) $\frac{\pi}{2}+\frac{\pi^2}{4}+\frac{\pi^3}{8}+\ldots=\sum_{n=1}^{\infty}\left(\frac{\pi}{2}\right)^{n-1}=$

d) $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ldots=$

e) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+2}}$

f) $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{2n+1}}$

g) $\sum_{n=0}^{\infty} \frac{2^{3n-1}}{7^{n+1}}$

Ex: Determine equivalent fraction:

$5.\overline{23}$

Def: A telescoping series is one in which the n[th] term can be expressed in the form

$a_n = b_n - b_{n+1}$

Convergence of a telescoping series:

If $\sum_{n=1}^{\infty} a_n$ is a telescoping series with $a_n = b_n - b_{n+1}$ then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $\{b_n\}$ converges. Furthermore, if $\{b_n\}$ converges to L, then $\sum_{n=1}^{\infty} a_n$ converges to L

Ex: Find the sum of the following:

a) $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 7n + 12}$$

b) $$\sum_{n=0}^{\infty} \left(e^{1/(n+3)} - e^{1/(n+2)} \right)$$

c) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

The test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex: Show that the following series is divergent.

a) $\sum_{n=1}^{\infty} \cos(n\pi)$

b) $\sum_{n=1}^{\infty} \frac{5n^3+2n-7}{2n^3+1}$

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series

$$\sum_{n=1}^{\infty} ca_n \quad \text{and} \quad \sum_{n=1}^{\infty} (a_n \pm b_n)$$

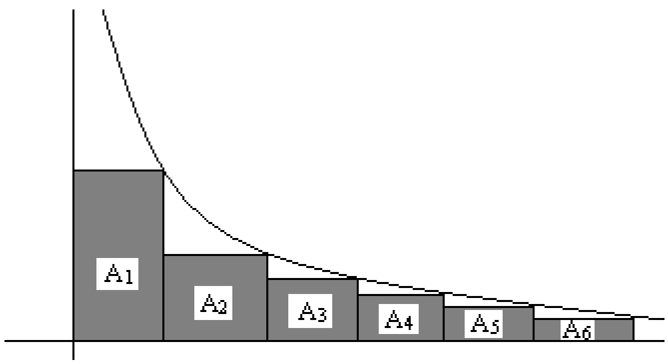
Ex: Find the sum of the series

a) $$\sum_{n=1}^{\infty} \left(\frac{1}{3^{n-1}} + \frac{1}{n(n+1)} \right)$$

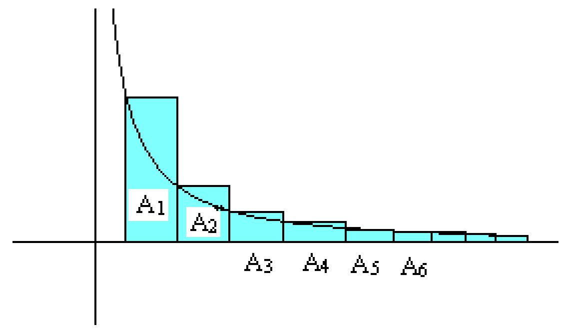
b) $$\sum_{n=0}^{\infty} \frac{\cos(n\pi) + 2^{n+1}}{5^n}$$

The Integral Test and Estimates of Sums

Given a series $\sum_{n=1}^{\infty} a_n$ and consider $f(x) = a_x$ $(i.e.\ replaced\ n\, by\, x)$



Case I



Case II

Introducing the test be given a series $\sum_{n=1}^{\infty} a_n$ and define a function $f(n) = a_n$

Case I: $\sum_{n=1}^{\infty} a_n \le \int_1^{\infty} f(x)dx$ ➔ If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Case II: $\sum_{n=1}^{\infty} a_n \ge \int_1^{\infty} f(x)dx$ ➔ $\int_1^{\infty} f(x)dx$ is divergent, and then $\sum_{n=1}^{\infty} a_n$ is divergent.

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Suppose that $a_n = f(n)$, where f is continuous, positive, decreasing function of x for all $x \ge N$ for some positive integer N. Then the series $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ both converge or both diverge.

Review p – test theorem:

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

P – test for series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent

Sol: Let $f(x) = \frac{1}{x^p} \Rightarrow \int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$ and it's divergent if $p \le 1$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and it's divergent if $p \le 1$

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{1}{(5n)^3}$

b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}}$

c) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Estimate the sum of a series:

Remainder Estimate for the integral test:

Suppose $f(n) = a_n$, where f is continuous, positive, decreasing function for $x \geq n$ and

$\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$

Ex: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms.
Estimate the error involved in this approximation.

b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Ex: Determine how many terms are needed to ensure the error within 0.005.

$$\sum_{n=1}^{\infty} \frac{1}{n\left[\ln(3n)\right]^3}$$