Alternating Series

Def: An alternating series is a series of the form $a_1 + a_2 - a_3 + a_4 - a_5 + ...$
Such as $1 + 2 - 3 + 4 - 5 + 6...$

The Alternating Series Test: Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - ... \quad (a_n > 0)$

Satisfies the following conditions:

i) $a_{n+1} \le a_n \Rightarrow$ Decreasing without negative sign.

ii) $\lim_{n\to\infty} a_n = 0$

Then the series is convergent.

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ this is an alternating series, but $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \ne 0$

c) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2^n}$

d) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+1}$; Let $a_n = \frac{n^2}{n^3+1}$; How do we know $a_n = \frac{n^2}{n^3+1}$ is decreasing, consider the following function $f(x)=\frac{x^2}{x^3+1} \Rightarrow f'(x)=\frac{x(2-x^3)}{(x^3+1)^2} \Rightarrow f'(x)<0$ for $x>\sqrt[3]{2}$ i.e.

$a_n = \frac{n^2}{n^3+1}$ is decreasing.

$\lim_{n\to\infty}\frac{n^2}{n^3+1}=0$. By the Alternating Series Test, $a_n = \frac{n^2}{n^3+1}$ is convergent.

e) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{3n+2}}$

f) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^4+5}$

<u>Estimating Sums</u>:

<u>Alternating Series Estimation Theorem</u>: If $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is the sum of an alternating series that satisfies

i) $0 \le a_{n+1} \le a_n$ and ii) $\lim_{n\to\infty} a_n = 0$ Then $|R_n| = |S - S_n| \le |a_{n+1}|$

Ex: Approximate the sum of the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ with an error of less than 0.01.

Ex: How many terms are needed in computing the sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^3+2n+4}$ to ensure its accuracy to 0.001