Section 11.6 Absolute convergence and The Ratio and Root Tests

Def: A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Ex: The series:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

b) $\sum_{n=1}^{\infty} \frac{\sin(5n-\pi)}{n^3}$

Def: A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{\sqrt{3n^7 + 2n^5 + 1}}$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Ex: Test for convergence / divergence:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{n^5+2n+7}$

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left[\ln(2n+1) - \ln(n+2)\right]$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{7^n+4^n}$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

The Ratio Test: Given a series $\sum_{n=1}^{\infty} a_n$

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1, \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$,

Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$

Ex: Test for the convergence:

a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{3^n}$

c) $$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

d) $$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$$

f) $$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$$

g) $$\sum_{n=1}^{\infty} \frac{4^n\, n!\,n!}{(2n)!}$$

h) $$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{(2\cdot 4\cdot 6\cdots(2n))\left[4^{n+2}+3\right]}$$

The Root Test: Given a series $\sum_{n=1}^{\infty} a_n$

i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (hence it's convergence)

ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex: Test the convergence /divergence of the following series:

a) $$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$$

b) $$\sum_{n=1}^{\infty}\left(\frac{7n^2+5n+4}{3n^2+2n+4}\right)^n$$

b) $$\sum_{n=1}^{\infty}\left(\frac{2n+3}{2n+5}\right)^{n^2}$$