Section 11.10 Taylor and Maclaurin Series

We have the power series as $\sum_{n=0}^{\infty} c_n (x-a)^n$. If we define a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Def: Let $f(x)$ has a power representation (expansion) at $x = a$. Where

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *where* $c_n = \dfrac{f^{(n)}(a)}{n!}$ is called a Taylor expansion of f(x) at $x = a$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: The Taylor polynomials of degree n at center $x = a$ $T_n(x,c) = \sum_{i=0}^{n} c_i (x-a)^i$

Def: Taylor expansion of f(x) at $x = 0$ is called Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex: Find the Taylor series of the following function at the center x = a.

a) $f(x)=\frac{1}{x}$ at $a=2$.

b) $f(x)=\sin(x)\ at\ a=\frac{\pi}{3}$

Ex: Find Maclaurin series of the following functions:

a) $f(x) = e^x$

b) $f(x) = \cos(x)$

c) $f(x) = \sin(x)$

Normally, we only interest at Taylor series up to certain degree n. So we $f(x) = T_n(x) + R_n(x)$, where $R_n(x)$ is the remainder (error) $R_n(x) = |f(x) - T_n(x)|$

Taylor's Theorem: If f is differentiable through order n + 1 in an open interval I containing a, then for each x in I, there exists a number c between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Taylor's Inequality: If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$

Note: $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$ for any real number x.

Ex: Find the Maclaurin series of the function $f(x) = \sin x$. Show that this series converges to $\sin x$ for all real x.

Ex: Find the Maclaurin series of the following functions:

a) $f(x) = x^3 \cos(7x^2)$

b) $f(x) = \dfrac{x^4}{e^{5x^3}}$

c) $f(x) = \dfrac{\sin(3x^3)}{3x^2}$

d) $f(x) = x^5 \tan^{-1}(2x^3)$

Ex: Using Maclaurin series to evaluate the following:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\pi^{2n+1}}{2^{2n+1}}$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\pi^{2n+1}}{9^n}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^n}{3^{n-1}}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{n/2}}{(2n+1)}$

Ex: a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Ex: Use power series to evaluate the following integrals:

a) $\int \frac{x^2}{3+5x^{15}}\,dx$

b) $\int x^2 \sin\left(x^{17}\right) dx$

c) $\int x^e e^{\sqrt{x}}\,dx$

Multiplication and Division of Power Series:

Ex: Find the first three nonzero terms in the Maclaurin series for

a) $f(x) = e^x \sin x$

b) $f(x) = \tan x$

Note: A famous Euler's formula (Euler identity)

Prove the Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$