Section 11.11 The Binomial Series

From algebra, how do we expand two – term – expression ➔ Pascal Triangle ➔ Binomial for positive integer exponent.

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

One of Newton's accomplishments was to extend the Binomial Theorem to the case in which k is no longer a positive integer. In this case for $(a+b)^k$ is no longer a finite sum; it becomes an infinite series. Let's exam the Maclaurin series of $(1+x)^k$.

The Binomial Series: If k is any real number and $|x| < 1$, then $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \text{ and } \binom{k}{0} = 1$$

Binomial Series: $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$ where we define $\binom{m}{1} = m$; $\binom{m}{2} = \frac{m(m-1)}{2!}$

And $\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$ for $k \geq 3$

Ex: Using Binomial series to expand: $f(x) = \frac{1}{(1+x)^2}$

b) $f(x)=\sqrt{1+x}$

c) $f(x)=\sqrt[3]{1+x}$

d) $f(x)=\dfrac{1}{\left(3+8x^3\right)^3}$

Section 11.12 Applications of Taylor Polynomials

<u>Approximating Functions by polynomials</u>

Suppose that $f(x)$ is equal to the sum of its Taylor series at a: $f(x)=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(a)}{n!}(x-a)^n$

So, we let $T_n(x)$ be the first nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a.

$$T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i \text{ and let the error be } R_n(x)=\sum_{i=n+1}^{\infty}\frac{f^{(i)}(a)}{i!}(x-a)^i=\left|f(x)-T_n(x)\right|$$

We have from Taylor Inequality

$$\left|R_n(x)\right|\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1} \text{ where } \left|f^{(n+1)}(x)\right|\le M$$

Ex: a) Approximate the function $f(x)=\sqrt[3]{x}$ by Taylor polynomial of degree 2 at $a=8$

b) How accurate is this approximation when $7 \le x \le 9$?

Ex: The third Maclaurin polynomial for $\sin x$ is given by: $\sin x \approx x - \frac{x^3}{3!}$. Use Taylor's Theorem to approximate $\sin(0.1)$ by $T_3(0.1)$ and determine the accuracy of the approximation:

Ex: Determine the degree of the Taylor polynomial $T_n(x)$ expanded about $a = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

Ex: Approximate $\sin 2^0$ accurate to four decimal places.

Ex: a) What is the maximum error possible in using the approximation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ where $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12^0$ corrects to six decimal places?

b) For what values of x is this approximation accurate to within 0.00005?