Section 3.11 ***Hyperbolic Functions***

Def:

Ex: Prove the following:

a) $\cosh^2(x) - \sinh^2(x) = 1$

b) $\sinh(2x) = 2\sinh(x)\cosh(x)$

c) $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Derivative of hyperbolic functions:

$\frac{d}{dx}\sinh x = \cosh x;$ $\qquad$ $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$

$\frac{d}{dx}\cosh x = \sinh x;$ $\qquad$ $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x;$ $\qquad$ $\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$

Ex: Differentiate the following functions:

a) $f(x) = e^{\sinh\left(3x^2+5\right)}\sqrt{\tanh^3(2x)+x}$

b) $f(x) = \dfrac{\operatorname{sech}\left(\sqrt{5x^3+1}\right)}{\ln\left(\sinh(3x)\right)}$

Inverse – Hyperbolic – functions:

$$y=\sinh(x) \Longrightarrow \sinh^{-1}(x)=?$$

$$y=\frac{e^x-e^{-x}}{2} \Rightarrow 2y=e^x-e^{-x} \Rightarrow 2y=e^x-\frac{1}{e^x}$$

$$e^{2x}-2ye^x-1=0 \Rightarrow e^x=\frac{2y\pm\sqrt{4y^2+4}}{2}=y\pm\sqrt{y^2+1}$$

$$x=\ln\left(y\pm\sqrt{y^2+1}\right) \Rightarrow \sinh^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$$

➔

$$\begin{cases} \sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right); & x\in R \\ \cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right); & x\geq 1 \\ \tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); & -1<x<1 \end{cases}$$

Note:

$$\begin{cases} y=\sinh^{-1}x \Leftrightarrow \sinh y=x \\ y=\cosh^{-1}x \Leftrightarrow \cosh y=x \\ y=\tanh^{-1}x \Leftrightarrow \tanh y=x \end{cases}$$

Derivative of inverse hyperbolic trig functions:

$$\begin{cases} \frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}; & \frac{d}{dx}\operatorname{csch}^{-1}x=-\frac{1}{|x|\sqrt{x^2+1}} \\ \frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}}; & \frac{d}{dx}\operatorname{sech}^{-1}x=-\frac{1}{x\sqrt{1-x^2}} \\ \frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}; & \frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^2} \end{cases}$$

Ex: Differentiate the following functions:

a) $f\left(x\right)=e^{\sinh\left(3x^2+1\right)}\tanh^{-1}\left(x^3+2\right)$

b) $f\left(x\right)=\sqrt{x}\sinh^{-1}\left(\sqrt{x}\right)$

c) $f\left(x\right)=x\tanh^{-1}x+\ln\left(\sqrt{1-x^2}\right)$