Section 4.3 **HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH**

$f'(x) > 0 \Rightarrow$ increase $\quad$ $f''(x) > 0 \Rightarrow$ *concave up*

$f'(x) < 0 \Rightarrow$ *decrease* $\quad$ $f''(x) < 0 \Rightarrow$ *concave down*

$f'(x) = 0 / DNE \Rightarrow$ critical points $\quad$ $f''(x) = 0 \Rightarrow$ Inflection point

Ex: Find where the function is increasing / decreasing / concave up / concave down.

a) $f(x) = x^3 + 5x^2 - 8x + 3$

b) $f(x) = x^{3/5}(x-3)$

Ex- Sketch a possible graph of the function satisfies:

a) $f(0)=2; f(2)=-1, f(-4)=0; f'(0)=f'(2)=0$

$f'(x)>0; 0<x<4; f'(x)<0; x<0; x>4$

b) $f'(2)=0; f'(0)=1,$

$f'(x)>0 \Leftrightarrow 0<x<2; \qquad f'(x)<0 \Leftrightarrow x>2, f''(x)<0 \Leftrightarrow 0<x<4$

$f''(x)>0 \Leftrightarrow x>4, \lim_{x\to\infty} f(x)=1, \; f(-x)=-f(x)$ for all x

Section 4.4 Indeterminate Forms and L'Hospital's Rule

In general, we are considering the limit of the form $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\frac{\infty}{\infty}$

<u>L'Hospital's Rule</u>: Suppose f and g are differentiable and $g'(x)\neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x)=0 \quad \text{and} \quad \lim_{x\to a} g(x)=0$$

or

$$\lim_{x\to a} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a} g(x)=\pm\infty$$

In other word, we are having the indeterminate forms such as $\frac{0}{0}$, or $\frac{\infty}{\infty}$

Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

<u>Proof:</u>

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}=\frac{\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a}}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{f(x)}{g(x)}$$

Ex: Evaluate the following limits:

a) $\lim\limits_{x\to 0}\frac{\sin(2x)}{x}$

b) $\lim\limits_{x\to\infty}\frac{e^x}{x^2}$

c) $\lim\limits_{x\to 0}\frac{e^x-1-x-(x^2/2)}{x^3}$

d) $\lim\limits_{x\to 0}\frac{\tan x-x}{x^3}$

Indeterminate Products: $0 \cdot (\pm\infty)$

Ex: Evaluate the following limits:

a) $\lim_{x \to 0^+} x \ln x$

b) $\lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2}\, dt$

c) $\lim_{x \to \infty} x \sin \frac{1}{x}$

Indeterminate Differences: $\infty - \infty$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit $\lim_{x \to a}\left[f(x) - g(x)\right]$

Ex: Evaluate

a) $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$

b) $\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

c) $\lim_{x \to \infty}\left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

Indeterminate Powers:

$$\lim_{x\to a}\left[f(x)\right]^{g(x)} \Leftrightarrow \begin{cases} 0^0 \\ \infty^0 \\ 1^\infty \end{cases}$$ Each of these three cases can be treated either by taking the natural logarithm i.e.

Let $y = f(x)^{g(x)} \Leftrightarrow \ln y = g(x)\ln f(x)$

Note: $$\lim_{x\to a}\ln f(x) = L \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a}\ln f(x)} = e^L$$

Ex: Evaluate the following limits

a) $\lim_{x\to 0^+}\left(1+\sin 4x\right)^{\cot x}$

b) $\lim_{x\to\infty} x^{1/x}$

Ex: Show that $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^{x} = e^{c}$ for any real number c.

Ex: Evaluate the following:

$$\lim_{x\to\infty}\left(\frac{3x-5}{3x+2}\right)^{2x+3}$$

<u>Cauchy's Mean Value Theorem</u>: Suppose that the functions f and g are continuous on [a, b] and differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there is a number $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$