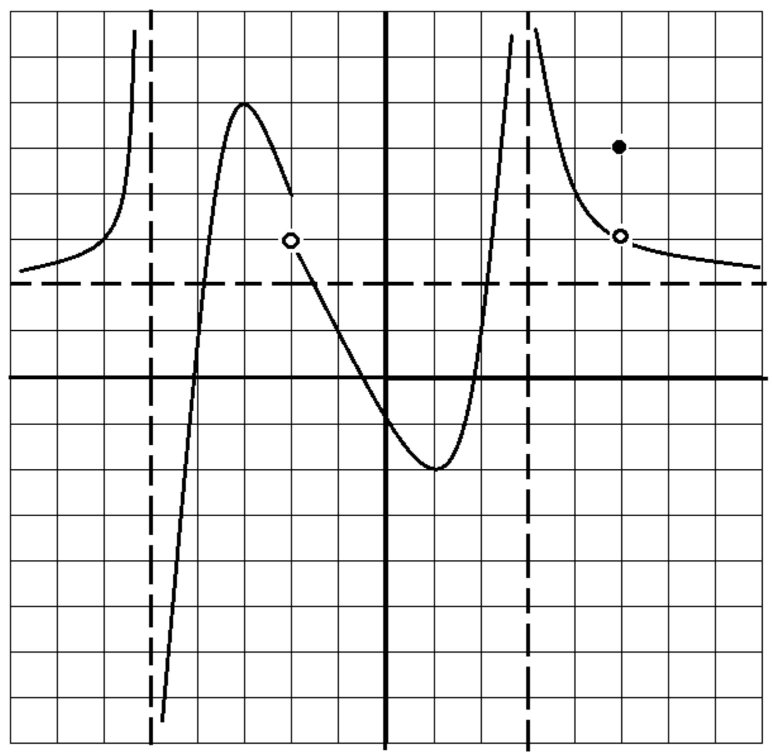
Section 2.1 – 2.2: ***The limit of a function***

Given a function f(x), we want to know what happens to f(x) when x behaves in a certain way, say x approaches to a certain number (either from one side or both sides) or when x approaches to (positive/negative) infinity.

Def: We write: $\lim_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, y approaches L"
i.e. if we make the values of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

Ex: Evaluate the limits of the following:



a) $\lim_{x \to 6} f(x) =$

b) $\lim_{x \to 4} f(x) =$

c) $\lim_{x \to -1} f(x) =$

d) $\lim_{x \to -2} f(x) =$

e) $\lim_{x \to -5} f(x) =$

f) $\lim_{x \to \infty} f(x) =$

g) $\lim_{x \to -\infty} f(x) =$

One side limits: $\lim_{x \to a^-} f(x)$; *for* $x < a$ *and* $\lim_{x \to a^+} f(x)$ *for* $x > a$

Ex: Let's explore this concept by examining the graph of the following function.

$$f(x)=\begin{cases} 3x-2\ ; & if\ \ x\le 3 \\ \dfrac{5}{x+2}+6; & if\ \ 3<x<5 \\ 3; & if\ \ x\ge 5 \end{cases}$$

Now look at the graph and answer the following questions:

a) $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ then determine $\lim_{x\to 3} f(x)=$

b) $\lim_{x\to 5^-} f(x)$ and $\lim_{x\to 5^+} f(x)$ then determine $\lim_{x\to 5} f(x)=$

c) $\lim_{x\to 4} f(x)$ and $\lim_{x\to 0} f(x)$

d) $\lim_{x\to \infty} f(x)$ *and* $\lim_{x\to -\infty} f(x)$

Ex: Sketch the graph of the following functions, and then determine their limits:

a) $\lim_{x \to \frac{\pi}{2}^-} \tan x$

b) $\lim_{x \to \pi} \csc x$

c) $\lim_{x \to 0^+} \ln(x)$

d) $\lim_{x \to \pi} \sin\left(\frac{x}{2}\right)$

e) $\lim_{x \to \infty} \tan^{-1}(x)$

f) $\lim_{x \to -\infty} \tan^{-1}(x)$

g) $\lim_{x \to 4} \frac{1}{(x-4)^7}$

h) $\lim_{x \to 4} \frac{1}{(x-4)^8}$

Definition: $\lim_{x \to a} f(x) = L$; iff $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Definition: Let f be a function defined on both sides of a, except possibly at a itself. Then $\lim_{x \to a} f(x) = \infty$ means that the values of f(x) can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

So far, if we know the graph of a function, we can easily determine the limit of a function, our main goal is what if we don't know the graph of a function, how do we determine the limit of a function algebraically?

- ⇨ 4 – Steps techniques:

Ex: Determine the limit of the following:

a) $\lim_{x \to 3} \dfrac{2x^2 - 5x - 3}{3x^2 - 4x - 15}$

b) $\lim_{x \to -2} \dfrac{\sqrt{5 - 2x} - 3}{x^2 - 4}$

c) $\lim_{t \to 0} e^{3\cos(2x)}$

d) $\lim_{x \to -1} \frac{\left|2x^2 + x - 1\right|}{3x^2 + x - 2}$

- Define vertical/horizontal asymptotes by taking limit. $\begin{cases} \lim_{x \to a} f(x) = \pm\infty (vertical) \\ \lim_{x \to \pm\infty} f(x) = a (horizontal) \end{cases}$

Ex: Sketch the graph of a function f(x) that satisfies the following conditions:

$\lim_{x \to 3^-} f(x) = 5$; $\lim_{x \to 3^+} f(x) = -2$; $\lim_{x \to \infty} f(x) = -2$; $\lim_{x \to -\infty} f(x) = -\infty$

$f(3) = 0$; $\lim_{x \to 7^+} f(x) = \infty$; $\lim_{x \to 7^-} f(x) = -\infty$

2.3 ***Calculating Limits Using the Limit Laws***

Limit laws: Suppose that c is a constant and the limits $\lim_{x \to a} f(x); \lim_{x \to a} g(x)$ exits. Then

1. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
3. $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
4. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}; \qquad \lim_{x \to a} g(x) \neq 0$
5. $\lim_{x \to a} [f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n$
6. $\lim_{x \to a} \mathrm{c} = \mathrm{c}$
7. $\lim_{x \to a} x^n = a^n$
8. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$
9. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
10. $\lim_{x \to a} b^{f(x)} = b^{\lim_{x \to a} f(x)}$

Theorem: If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Ex: Evaluate the following limits and justify each step

a) $\lim_{x \to 3} e^{\frac{2x^2 - 5x - 3}{3x^2 - 5x - 12}}$

b) $\lim_{x \to -3} \sqrt[3]{\dfrac{2x^2 + x - 15}{x^2 + 7x + 12}}$

c) $\lim_{h\to 0}\frac{\frac{1}{h+5}-\frac{1}{5}}{h}$

d) $\lim_{h\to 0}\frac{\frac{1}{\sqrt{2h+9}}-\frac{1}{3}}{h}$

e) $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$

f) $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} 3x^2+x-7; & if \quad x<1 \\ 5\sin\left(\frac{\pi x}{2}\right)-7x-1; & if \quad x\geq 1 \end{cases}$

Theorem: If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exit as x approaches a,

then $$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

The Squeeze theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} g(x) = L$$

Ex: Evaluate the following limits:

a) $$\lim_{x\to 0} x^6 \sin\left(\frac{3}{2x^2}\right)$$

b) If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x\to 1} f(x)$.

c) *If* $\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$ *for x close to zero. Evaluate* $\lim_{x\to 0} \frac{1-\cos x}{x^2}$