Section 7.4 ***Derivatives of a Transform***

Given $F(s) = L(f(t)) \Rightarrow \frac{d}{ds}F(s) =$

Theorem: If $F(s) = L(f(t)) \Rightarrow \frac{d^n}{ds^n}(F(s)) = (-1)^n L(t^n f(t))$

(i.e. Given *If* $L(f(t)) = F(s) \Rightarrow$ *Then* $L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(F(s))$)

Ex: Determine the Laplace Transform of the following functions:

a) $L(t^3 e^{2t})$

b) $L(t\cos(2t))$

c) $L\left(t\sin\left(3t\right)\right) =$

Note: $L\left(t\sin\left(kt\right)\right)$

d) $L\left(t^2\sin\left(3t\right)\right)$

Ex: Solve: $y''+16y=\cos(4t);\ y(0)=0;\ y'(0)=1$

The Convolution Integral

The question: If $F(s)=H(s)G(s)$ what is $L^{-1}(F(s))=?=L^{-1}(H(s))L^{-1}(G(s))$

Def: Suppose that f(t) and g(t) are continuous on the interval $[0,b]$, then for $t\in(0,b]$, the convolution product

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau$$

Ex: Determine the convolution of the following:

a) $f(t)=t \text{ and } g(t)=e^t$

b) $f(t)=\sin(t) \text{ and } g(t)=\cos(t)$

Property of Convolution:

1. $f*g=g*f$
2. $f*(g*h)=(f*g)*h$
3. $f*(g+h)=f*g+f*h$

__Theorem:__ (The Convolution Theorem)

Given: $f(t)$ *and* $g(t)\in E(0,\infty)$ with $F(s)=L(f(t))$ *and* $G(s)=L(g(t))$

Then $L(f*g)=L(f)L(g)=F(s)G(s)$.

Conversely: $L^{-1}(F(s)G(s))=(f*g)(t)$

Note: $L^{-1}\left[F_1(s)F_2(s)...F_n(s)\right]=(f_1*f_2*f_3*...*f_n)(t)$

Ex: Find

a) $L^{-1}\left(\dfrac{1}{s^2(s-1)}\right)$

b) $L^{-1}\left(\dfrac{2s}{(s^2+1)^2}\right)$

Ex: Verify: $L^{-1}\left(\dfrac{1}{s^2+s-6}\right)$ by

a) $L^{-1}\left(\dfrac{1}{s+3}\right) * L^{-1}\left(\dfrac{1}{s-2}\right)$

b) $L^{-1}\left(\dfrac{1}{s^2+s-6}\right) = L^{-1}\left(\dfrac{A}{s+3} + \dfrac{B}{s-2}\right)$

Ex: Solve the DE:

a) $y''+2y'+2y=\sin(3t);\ y(0)=0;\ y'(0)=0$

b) $y'' + 4y' + 13y = \cos(3t);\ y(0) = 1;\ y'(0) = 2$