Section 8.3 ***Non-Homogenous Linear Systems:***

Undetermined coefficients:

Ex: Solve $\dfrac{\overrightarrow{dx}}{dt}=\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}\vec{x}+\begin{bmatrix}-8\\ 3\end{bmatrix}$ $over\ (-\infty,\infty)$

Sol:

Ex: $\vec{x}\,' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6t \\ -10t+4 \end{bmatrix}$ *over* $(-\infty, \infty)$

Variation of Parameters: Given $\overrightarrow{x'(t)} = A\overrightarrow{x(t)} + \overrightarrow{F(t)}$

Let $\overrightarrow{x_1(t)}, \overrightarrow{x_2(t)}, ..., \overrightarrow{x_n(t)}$ be a fundamental set of solutions of the homogeneous system $\overrightarrow{x'(t)} = A\overrightarrow{x(t)}$ over an interval I.

$$\overrightarrow{x_h(t)} = C_1\overrightarrow{x_1} + C_2\overrightarrow{x_2} + ... + C_n\overrightarrow{x_n} = C_1\begin{bmatrix} x_{11} \\ x_{21} \\ ... \\ x_{n1} \end{bmatrix} + C_2\begin{bmatrix} x_{12} \\ x_{22} \\ ... \\ x_{n2} \end{bmatrix} + ... + C_n\begin{bmatrix} x_{1n} \\ x_{2n} \\ ... \\ x_{nn} \end{bmatrix} = \begin{bmatrix} C_1x_{11} + C_2x_{12} + ... + C_nx_{1n} \\ C_1x_{21} + C_2x_{22} + ... + C_nx_{2n} \\ .... \\ C_1x_{n1} + C_2x_{n2} + ... + C_nx_{nn} \end{bmatrix}$$

$$\overrightarrow{x(t)} = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{nn} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ ... \\ C_n \end{bmatrix}$$

Define $\Phi(t) = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{nn} \end{bmatrix}$, since column vector is linearly independent. ➔ $\Phi(t)$ is nonsingularity.

We call $\Phi(t)$ is a fundamental matrix, $\overrightarrow{x'(t)} = A\overrightarrow{x(t)} \Longrightarrow \Phi'(t) = A\Phi(t)$

$\det(\Phi(t)) \neq 0 \Rightarrow \Phi^{-1}(t)$ *exists for* $t \in I$

Variation of Parameters: Let $\overrightarrow{U(t)} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ ... \\ u_n(t) \end{bmatrix}$ *such that* $\overrightarrow{x_p(t)} = \Phi(t)\overrightarrow{U(t)}$ be a solution for

$\overrightarrow{x'(t)} = A\overrightarrow{x(t)} + \overrightarrow{F(t)}$

Product Rule: $\overrightarrow{x_p'(t)} = \Phi'(t)\overrightarrow{U(t)} + \Phi(t)\overrightarrow{U'(t)} = A\Phi(t)\overrightarrow{U(t)} + \Phi(t)\overrightarrow{U'(t)} = A\Phi(t)\overrightarrow{U(t)} + \overrightarrow{F(t)}$

$$\Phi(t)\overrightarrow{U'(t)} = \overrightarrow{F(t)} \Rightarrow \Phi^{-1}(t)\Phi(t)\overrightarrow{U'(t)} = \Phi^{-1}(t)\overrightarrow{F(t)}$$

$$U(t) = \int \Phi^{-1}(t)\overrightarrow{F(t)}dt$$

$$\overrightarrow{x_p(t)} = \Phi(t)\int \Phi^{-1}(t)\overrightarrow{F(t)}dt$$

$$General\ sol: \overrightarrow{x(t)} = \Phi(t)\overrightarrow{C} + \Phi(t)\int \Phi^{-1}(t)\overrightarrow{F(t)}dt$$

Ex: Solve: $\overrightarrow{x'(t)} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \overrightarrow{x(t)} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$