Review: First shift theorem: $L\left(e^{at} f(t)\right) = F(s-a)$ *given* $L\left(f(t)\right) = F(s)$

$L^{-1}\left(F(s-a)\right) = e^{at} f(t)$ *given* $F^{-1}\left(F(s)\right) = f(t)$

Solve: $y'' + 2y' - 3y = 26e^{2t} \cos t;\ \ y(0) = 1;\ y'(0) = 0$

The Unit Step Function

Def: $u_a(t)=\begin{cases}0; & 0\le t<a\\ 1; & t\ge a\end{cases}$ ➔ To turn on a function f(t) at t = a and off at t = b;

$h(t)=f(t)\left[u_a(t)-u_b(t)\right]$

Ex: The voltage in a circuit is given by $E(t)=\begin{cases}20t; & 0\le t<5\\ 0; & t\ge a\end{cases}$

this can be written by $E(t)=20t-20tu_5(t)$

Ex: $f(t)=\begin{cases}\cos t & 0\le t<2\pi\\ e^{t-2}; & 2\pi\le t<20\\ t+3; & t\ge 20\end{cases}\Rightarrow$

The Second Shifting Theorem (the translation of t – axis)

If $L(f(t))=F(s)$ *then* $L(f(t-a)u_a(t))=e^{-as}L(f(t))=e^{-as}F(s)$

Conversely, $L^{-1}\left(e^{-as}F(s)\right)=u_a(t)f(t-a)$

Proof:

Ex: Evaluate the following:

a) $L\left(u_2(t)(t-2)^6\right)$

b) $L\left(u_\pi(t)\cos(5t-5\pi)\right)$

c) $L\left(u_{2\pi}(t)e^{-2t}\cos(3t-6\pi)\right)$

d) $L\left(u_4(t)e^{2t}\right)$

Note:

$$L\left(u_a(t)\right) = L\left(u_a(t) f(t)\right); \ \textit{where } f(t) = 1 \Rightarrow f(t-a) = 1$$

$$L\left(u_a(t)\right) = e^{-as} L(1) = e^{-as}\left(\frac{1}{s}\right) = \frac{e^{-as}}{s}$$

d) $L\left(2 - 3u_2(t) + u_3(t)\right)$

e) $f(t) = \begin{cases} 0; & 0 \le t < 1 \\ t-1; & 1 \le t < 2 \\ 1; & t \ge 2 \end{cases}$

Note: What if it does not match the shifting: $L\left(u_a(t)f(t)\right)=e^{-as}L\left(f(t+a)\right)$

f) $f(t)=u_3(t)t^2$

g) $f(t)=u_3(t)\sin(5t)$

h) $f(t)=u_3(t)e^{5t}$

i) $f(t)=u_3(t)e^{5t}\cos(2t)$

j) $f(t)=\begin{cases} 3; \ if \ \ 0 \le t < 2 \\ e^{2t}; \quad if \ 2 \le t < 5; \\ t^2+2t-3; \ \ t \ge 5 \end{cases}$

k) $f(t)=\begin{cases} 2 \ \ 0 \le t < 3 \\ t \ \ \ 3 \le t < 4 \\ \cos(2t) \quad t \ge 4 \end{cases}$

Note: $L\left(u_a(t)f(t-a)\right)=e^{-sa}L\left(f(t)\right)$

$L^{-1}\left(e^{-as}F(s)\right)=u_a(t)f(t-a) \text{ where } L\left(f(t)\right)=F(s)$

Ex: Find inverse Laplace transform:

a) $F(s)=\dfrac{e^{-3s}s}{s^2+7}$

b) $F(s)=\dfrac{e^{-4s}}{(s+5)^4}$

c) $F(s)=\dfrac{e^{-4s}}{s^2+4s+9}$

c) $F(s)=\dfrac{1-e^{-s}}{s^2}$

e) $$F(s) = \frac{e^{-3s}(s-4)}{s^2 - 4s + 5}$$

e) $$F(s) = \frac{(s-5)e^{-4s}}{s^2 - 2s + 6}$$

f) $$F(s) = \frac{e^{-2s}(17s+50)}{(s+1)(s^2+6s+16)}$$

Ex: Solve the following DE: $y''+2y'+5y=f(t);\ y(0)=0;\ y'(0)=0$

$f(t)=10-20u_4(t)+10u_8(t)=10[1-2u_4(t)+u_8(t)];$

Ex: Solve: $y''+16y=f(t)=\begin{cases}\cos(4t);\ 0\le t<\pi\\ 0;\ \ t\ge\pi\end{cases};\ \ y(0)=0;\ y'(0)=1$