Annihilator:

Convert a non-homogeneous DE to Homogeneous DE:

The problems that we have looked at are of the form:
$L(D)y = g(x)$, and $Q(D)F = 0$ Where Q(D) is a polynomial differential operator. Any polynomial differential operator Q (D) that satisfies $Q(D)F = 0$ is said to annihilate F(x). The ***polynomial differential operator of lowest order that satisfies equation*** $Q(D)F = 0$ ***is called the annihilator of F.***
So, if we apply Q(D) to both sides of $L(D)y = g(x)$, then we have $Q(D)L(D)y = Q(D)g(x) = 0$.
This is a constant coefficient homogeneous linear DE and therefore can be solved using the technique of the previous sections.

Def: A linear differential operator Q(D) is said to annihilate a function f(x) if $Q(D)f(x) = D(f(x)) = 0$ for all x. That is, Q annihilates f is f is a solution to the homogeneous linear differential equation on $(-\infty, \infty)$

Ex: For DE $y'' - 4y' + 20y = 0 \Rightarrow Q(D) = D^2 - 4D + 20$ is an annihilator of $e^{2x}\sin 4x$ and $e^{2x}\cos 4x$

For $f(x) = k \Rightarrow Q(D) = D$

For $f(x) = x^n \Rightarrow Q(D) = D^{n+1}$

For $f(x) = e^{ax} \Rightarrow Q(D) = (D - a)$

For $f(x) = xe^{ax} \Rightarrow Q(D) = (D - a)^2$

For $f(x) = \sin bx;\ or\ \cos bx \Rightarrow Q(D) = D^2 + b^2$

For $f(x) = x\sin bx,\ or\ x\cos bx \Rightarrow Q(D) = (D^2 + b^2)^2$

For $f(x) = e^{ax}\sin bx,\ or\ e^{ax}\cos bx \Rightarrow Q(D) = (D - a)^2 + b^2$

<u>In general</u>: $(D-r)^m$, where m is a positive integer, annihilates each of the following functions:

$e^{rx}, xe^{rx}, x^2e^{rx}, ..., x^{m-1}e^{rx}$

and $\left[(D-\alpha)^2+\beta^2\right]^m$ annihilates each of the following functions:

$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, ..., x^{m-1}e^{\alpha x}\cos\beta x$

$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, ..., x^{m-1}e^{\alpha x}\sin\beta x,$

Note: (Annihilator of a sum of functions) Let $Q_g(D)$ *and* $Q_h(D)$ be annihilator of g(x) and h(x) respectively. Then $Q(D)=Q_g(D)Q_h(D)$ is an annihilator for $f(x)=g(x)+h(x)$

Ex: Find an annihilator of the following functions:

a) $g(x)=3x^4-2x^3-4x-1$

b) $g(x)=7e^{-4x}$

c) $g(x)=8\sin(3x)$

d) $g(x)=e^{3x}\cos(4x)$

e) $g(x)=e^{-2x}\sin(5x)+4x^3+5x+2$

f) $g(x) = x^3 e^{3x} + 2\sin(5x)$

g) $g(x) = x^2 e^{3x} \cos(3x)$

h) $g(x) = 5\cos^2(3x)$

So, how do we use Annihilator to solve $L(D)y = g(x)$.

Find an annihilator for the non-homogeneous g(x), say $Q(D)$. Solve for homogeneous solution first then apply $Q(D)$ to both sides of the $L(D)y = g(x) \Rightarrow Q(D)L(D)y = Q(D)g(x) = 0$

Ex: Use the annihilator method to solve the following:

a) $(D-3)^2 y = 7e^{3x}$

b) $y'' + y = e^{2x} + 1$

c) $y'' - y = e^{-2x} \sin x$

d) $(D+1)(D^2+9)y = 4xe^{-x} + 5e^{2x}\cos 3x$

e) $y'' + y' + \frac{1}{4}y = e^{x}\left(\sin(3x) - \cos(3x)\right)$

f) $y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x}$

g) $\left(D^2 + D - 2\right)y = 4\cos x - 2\sin x;\ y(0) = -1,\ y'(0) = 4$

h) $\left(D^2+2D-3\right)y=\sin^2\left(x\right)$

f) $y''+5y'-6y=10e^{2x};\; y\left(0\right)=1,\; y'\left(0\right)=1$