4.7 A Differential Equation with Non – Constant Coefficients.

Def: A DE of the form: $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_{n-1} x \frac{dy}{dx} + a_n y = 0$ where $a_1, a_2, ..., a_n$ are constants, is called a Cauchy – Euler equation.

For Second Order Homogeneous Cauchy DE: $x^2 y'' + a_1 xy' + a_2 y = 0$

Case1: Two real distinct roots: $y = C_1 x^{r_1} + C_2 x^{r_2}$

Ex: Solve: $x^2 y'' - xy' - 8y = 0;$

Case 2: Repeated roots: $r = r_1 = r_1 \Rightarrow y = C_1 x^{r_1} + C_2 x^{r_2} \ln\left(x\right)$

Ex: Solve: $4x^2 y'' + 8xy' + y = 0$

Case 3: Two complex roots: $r = \alpha \pm \beta i \Rightarrow y = x^{\alpha}\left(C_1 \cos\left(\beta \ln x\right) + C_2 \sin\left(\beta \ln x\right)\right)$

Ex: Solve: $x^2 y'' - 3xy' + 13y = 0;\ y(1) = 2;\ y'(1) = -5$

Now, consider non-homogeneous Cauchy DE.

$$x^2 y'' + a_1 x y' + a_0 y = g(x)$$

$$y'' + \frac{a_1}{x} y' + \frac{a_0}{x^2} y = \frac{g(x)}{x^2}$$

We define $y'' + P(x) y' + Q(x) y = f(x)$

After find homogeneous solution for $x^2 y'' + a_1 x y' + a_0 y = 0 \Rightarrow y_h = C_1 y_1 + C_2 y_2$.

Using Variation of Parameters to solve for the non-homogenous: $y = u_1 y_1 + u_2 y_2$

Then $u_1' = -\frac{y_2 f(x)}{W}$; $u_2' = \frac{y_1 f(x)}{W}$; where $f(x) = x^{-2} g(x)$

Ex: Solve $x^2 y - 3xy' + 3y = 2x^4 e^x$

Ex: $x^2 y'' - 3xy' + 4y = x^2 \ln x$

4.9 Solving Systems of Linear DEs by Elimination:

Definition: $$\begin{cases} \dfrac{dy_1}{dx} = a_{11}(x)y_1 + a_{12}(x)y_2 + ... + a_{1n}(x)y_n + b_1(x) \\ \dfrac{dy_2}{dx} = a_{21}(x)y_1 + a_{22}(x)y_2 + ... + a_{2n}(x)y_n + b_2(x) \\ ... \\ \dfrac{dy_n}{dx} = a_{n1}(x)y_1 + a_{n2}(x)y_2 + ... + a_{nn}(x)y_n + b_n(x) \end{cases} \Rightarrow$$

$$\begin{bmatrix} y_1' \\ y_2' \\ ... \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix} \Rightarrow \overrightarrow{y'} = A\overrightarrow{y} + \vec{b}$$

Ex: The system:

$$\begin{cases} y_1' = 3y_1 + 2xy_2 + x^4 \\ y_2' = y_1 \sin x + x^3 y_2 - \ln x \end{cases} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3 & 2x \\ \sin x & x^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x^4 \\ -\ln x \end{bmatrix}$$

Ex: Verify that $y_1 = -2e^{5x} + 4e^{-x}$ *and* $y_2 = e^{5x} + e^{-x}$ is a solution to the system of linear equations: $\begin{cases} y_1' = y_1 - 8y_2 \\ y_2' = -y_1 + 3y_2 \end{cases}$

Solve by substitution and elimination:

Ex: Solve the system:

a) $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 2y_1 - 2y_2 \end{cases}$

Ex: Solve

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = 2x_1 - 2x_2 \end{cases} ;\ x_1(0) = 1;\ x_2(0) = 0$$

b) $\begin{cases} Dx + (D+2)y = 0 \\ (D-3)x + 2y = 0 \end{cases}$

c) $\begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = -x_1 + 4x_2 \end{cases}$; $x_1(0) = 1$; $x_2(0) = 3$

d) $\begin{cases} x' - 4x + y'' = t^2 \\ x' + x + y' = 0 \end{cases}$

d) $$\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t \\ -\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0 \end{cases}$$