Section 2.4 ***Exact DE***

For the next technique it is best to consider first-order DE written in differential form: $M(x,y)dx + N(x,y)dy = 0$ where M and N are given functions, assumed to be sufficiently smooth. The method that we will consider is based on the idea of a differential.

Def: The differential form $M(x,y)dx + N(x,y)dy$ is said to be ***exact*** in a rectangle R if there is a function $\phi(x,y)$ (we call this as a ***potential function***) such that $M(x,y)dx + N(x,y)dy = 0$ and $\frac{\partial \phi}{\partial x} = M; \frac{\partial \phi}{\partial y} = N$ for all x,y in the rectangle R.

Ex: a) $ydx + xdy = 0 \Rightarrow M = y, N = x$ is exact because if we look at $\phi(x,y) = xy \Rightarrow \frac{\partial \phi}{\partial x} = y = N; \frac{\partial \phi}{\partial y} = x = M$

b) $2x \sin y dx + x^2 \cos y dy = 0 \Rightarrow M = 2x \sin y; N = x^2 \cos y;$
Consider $\phi(x,y) = x^2 \sin y$

Theorem: The general solution to an exact equation $M(x,y)dx + N(x,y)dy = 0$ is defined implicitly by $\phi(x,y) = c$ where $\frac{\partial \phi}{\partial x} = M; \frac{\partial \phi}{\partial y} = N$ and c is an arbitrary constant.

Proof: Let rewrite $M(x,y)dx + N(x,y)dy = 0$ as $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ being an exact DE, we can replace $\frac{\partial \phi}{\partial x} = M; \frac{\partial \phi}{\partial y} = N$ into the equation such as $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{d}{dx}\phi(x,y) = 0 \Rightarrow \phi(x,y) = c$

Remarks:

1. The potential function ϕ is a function of two variables x and y, and we interpret the relationship $\phi(x,y) = c$ as ***defining y implicitly as a function of x.*** The preceding theorem states that this relationship defines the general solution to the DE for which ϕ is a potential function.

2. Geometrically, the theorem says that the solution curves of an exact DE are the family of curves $\phi(x,y) = c$. These are called the level curves of the function $\phi(x,y)$.

Now, we are ready to solve exact DE. The questions are of course:

a) How can we tell whether a given DE is exact?
b) If it's an exact DE, how do we find a potential function?

<u>Theorem</u>: (***Test for Exactness***): Let M, N and their first partial derivatives M_y and N_x be continuous in a (simply connected) region R of xy-plane. Then the DE $M(x,y)dx + N(x,y)dy = 0$ is exact for all x,y in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Ex: Determine whether the given DE is exact.

a) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$

b) $[1 + \ln(xy)]dx + (x / y)dy = 0$

c) $x^2 y dx - (xy^2 + y^3)dy = 0$

Method for solving exact equations:

a) If $M(x,y)dx + N(x,y)dy = 0$ is exact, then let $\frac{\partial \phi}{\partial x} = M$. Integrate this last equation with respect to x to get.

$\phi(x,y) = \int M(x,y)dx + g(y)$ (*)

b) To determine $g(y)$, take the partial derivative with respect to y of both sides of equation (*) and substitute N for $\frac{\partial \phi}{\partial y}$. We can now solve for $g'(y)$

c) Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (*) to get $\phi(x,y)$

d) The solution to $M(x,y)dx + N(x,y)dy = 0$ is given implicitly by $\phi(x,y) = c$

Ex: Solve the following DE:

a) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$

b) $2xe^{y}\,dx + \left(x^2e^{y} + \cos y\right)dy = 0$

c) $\left[\sin(xy) + xy\cos(xy) + 2x\right]dx + \left[x^2\cos(xy) + 2y\right]dy = 0$

Now we develop another technique to handle those DE that are almost exact, but not quite.

Integrating Factors for exact DE

Def: A nonzero function $I(x,y)$ is called an integrating factor for $M(x,y)dx+N(x,y)dy=0$ if the DE $I(x,y)M(x,y)dx+I(x,y)N(x,y)dy=0$ is exact.

Ex: Show that $I=x^2y$ is an integrating factor for the DE $\left(3y^2+5x^2y\right)dx+\left(3xy+2x^3\right)dy=0$

Theorem: Consider the DE $M(x,y)dx+N(x,y)dy=0$.

1. There exists an integrating factor that depends only on x if and only if $\left(M_y-N_x\right)/N=f(x)$ a function of x only.

   In such a case, an integrating factor is $I(x)=e^{\int f(x)dx}$

2. There exists an integrating factor that depends only on y if and only if $\left(M_y-N_x\right)/M=g(y)$, a function of y only.

   In such a case, an integrating factor is $I(y)=e^{-\int f(y)dy}$

Proof: We will prove only (1). Suppose first that $I(x)=e^{\int f(x)dx}$ is an integrating factor of $M(x,y)dx+N(x,y)dy=0$. $\frac{\partial I}{\partial y}=0$. So from the previous theorem, we have $N\frac{\partial I}{\partial x}-M\frac{\partial I}{\partial y}=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)I$, we have $I(x)=e^{\int f(x)dx}$ is a solution to $\frac{dI}{dx}N=\left(M_y-N_x\right)I\Rightarrow\frac{1}{I}\frac{dI}{dx}=\frac{M_y-N_x}{N}$. Since, by assumption, I is a function of x only, it follows that the left-hand side of this expression depends only on x and hence also the right hand side.

Conversely, suppose that $\left(M_y-N_x\right)/N=f(x)$, a function of x only. Then, dividing $N\frac{\partial I}{\partial x}-M\frac{\partial I}{\partial y}=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)I$ by N, it follows that I is an integrating factor for $M(x,y)dx+N(x,y)dy=0$ if and only if it is a solution to $\frac{\partial I}{\partial x}-\frac{M}{N}\frac{\partial I}{\partial y}=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\frac{I}{N}=If(x)\Leftarrow$ we want to show that this equation must have a solution I which depends only on x. We assume that $I=I(x)\Rightarrow\frac{\partial I}{\partial y}=0\Rightarrow\frac{\partial I}{\partial x}=If(x)\Leftrightarrow\frac{dI}{dx}=If(x)\Leftrightarrow\frac{dI}{f(x)}=Idx\Leftrightarrow I(x)=e^{\int f(x)dx}$

Ex: Solve $(2x - y^2)dx + xydy = 0, \; x > 0$

b) $(2xy + 2x^3y - 1)dx + (1 + x^2)^2 \, dy = 0$