Introduction of DE:

Def: An equation containing the derivatives of one or more unknown functions (dependent variables), with respect to or more independent variables, is said to be a differential equation (DE)

Definition of solution of an DE: Any function Phi, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth – order ordinary DE reduces the equation to an identity, is said to be a solution of the equation on the interval.

Ex: Verify that if given function is a solution of a DE $C_1; C_2; ...; C_n$ are arbitrary constants.

a) $y(x) = C_1 e^{-5x} + C_2 e^{3x} \quad for \ y'' + 2y' - 15y = 0$

b) $y = C_1 x^{-3} + C_2 x^{-1} \quad for \ x^2 y'' + 5xy' + 3y = 0$

c) $y(x) = C_1 x^{1/2} + 3x^2 \quad for \quad x^2 y'' - xy' + y = 9x^2$

d) $e^{xy} - x = C; \; y' = \dfrac{1 - ye^{xy}}{xe^{xy}}$

Normal form: $\dfrac{d^n y}{dx^n} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$

Definition Implicit Solution of an ODE:

A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation on an interval I, provided that there exists at least one function phi that satisfies the relation as well as the differential equation on I.

Ex: Verify that the relation $\sin(xy)+y^2-x=0$ defines a solution to $\frac{dy}{dx}=\frac{1-y\cos(xy)}{x\cos(xy)+2y}$

Initial – Value Problems (IVP)

Solve $\frac{d^n y}{dx^n}=f\left(x,y,y',y'',\ldots,y^{(n-1)}\right)$ subject to $y(x_0)=y_0;\ y'(x_0)=y_1,\ y''(x_0)=y_2,\ldots,y^{(n-1)}(x_0)=y_{n-1}$

Ex: Solve: $y''=18\cos(3x);\ y(0)=1,\ y'(0)=4$

Ex: Verify that $y = e^{-3x}\left(A\cos(2x) + B\sin(2x)\right)$ for $y'' + 6y' + 13y = 0$, Then determine the constants for

$y(0) = 2 \ and \ y'(0) = 2$

Basic terminology:

Nth – order of a DE:

Linear / Non-linear:

Homogeneous DE and Non – Homogeneous DE:

Existence of a Unique Solution: Given 1^{st} – order – DE: $\frac{dy}{dx} = f(x,y)$

Let R be a rectangular region in the xy – plane define by $a \le x \le b,\ c \le y \le d$ that contain the point (x_0, y_0). If $f(x,y)$ *and* $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : (x_0 - h, x_0 + h);\ h > 0$ contained in $[a,b]$, such that the initial – value problem has a unique solution for x in I_0

Ex: Prove that the initial – value problem

$\frac{dy}{dx} = 3xy^{1/3};\ y(0) = a$ has a unique solution whenever $a \ne 0$

Ex: Determine the region R where the IVP has a unique solution:

a) $\frac{dy}{dx} = \sqrt{x^2 + y^2}\quad y(0) = 2$

b) $\frac{dy}{dx} = (3x + y)^{2/3}$; $y(1) = -3$

c) $\frac{dy}{dx} = x^2 - xy^3$; $y(1) = 6$,

d) $\frac{dy}{dx} = 3xy^{1/3}$; $y(0) = 0$

Section 1.3 Differential Equations as Mathematic Models

Ex1: The number of bacteria in a culture grows at a rate that is proportional to the number present. If there were 10 bacteria in the culture. If the double time of the culture is 3 hours, find the number of bacteria that were present after 24 hours?

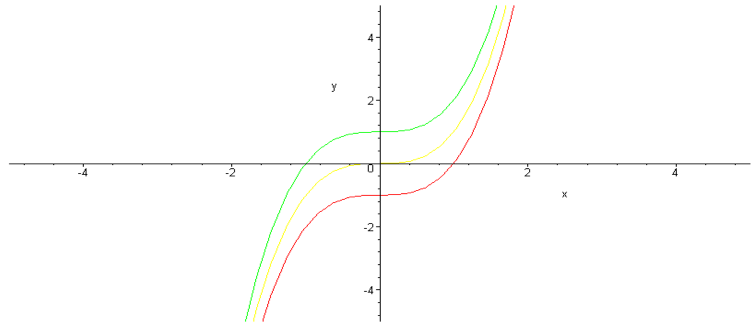
Ex2: Of the 1500 passengers, crew, and staff that board a cruise ship, 5 have the flu. After one day of sailing, the number of infected people has risen to 10. Assuming that the rate at which the flu virus spreads is proportional to the product of the number of infected individuals and the number not yet infected, determine how many people will have the flu at the end of 14 – day cruise.

Ex3: A tank contains 600 gal of water in which there is dissolved 4lbs of salt. A solution containing ½ lb of salt flows into the tank at the rate of 5gal/min, and the well-stirred mixture flows out at a rate of 3 gal/min. Determine the concentration of salt in the tank after 1 hour.

Section 2.1 *The Geometry of First – Order DE* $\frac{dy}{dx} = f(x, y)$

First order linear DE, with the RHS contains only independent variable.

Ex: Solve $\frac{dy}{dx} = x^2 \Rightarrow y = \frac{1}{3}x^3 + c \Rightarrow$ Graph for this solution is of course depending on the c-value.



Existence and Uniqueness Theorem (Revisited)

Let $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$ where f(x,y) is continuous on the rectangle: $R = \{(x, y) | a \le x \le b, c \le y \le d\}$. Suppose further that $\frac{\partial f}{\partial y}$ is continuous in R. Then for any interior point (x_0, y_0) in the rectangle R, there exists an interval I containing x_0 such that the IVP has a unique solution for x in I.

NOTE: From geometric viewpoint, if f(x,y) satisfies the hypotheses of the existence and uniqueness theorem in a region R of the xy-plane then throughout the region the solution curves of the DE $\frac{dy}{dx} = f(x, y)$ cannot intersect. For if two solution curves did intersect at (x_0, y_0) in R, then that would imply that there was more than one solution to the IVP, which would contradict the existence and uniqueness theorem.

Slope Fields (Isoclines): $\frac{dy}{dx} = f(x, y) \Rightarrow$ where $f(x, y)$ gives the slope of the tangle line to the solution curves of this DE at point (x,y) . So let $f(x, y)$=k be a constant for certain slope.

Ex: Sketch the slope field of the following: (autonomous / non-autonomous)

a) $\frac{dy}{dx} = \sin y$

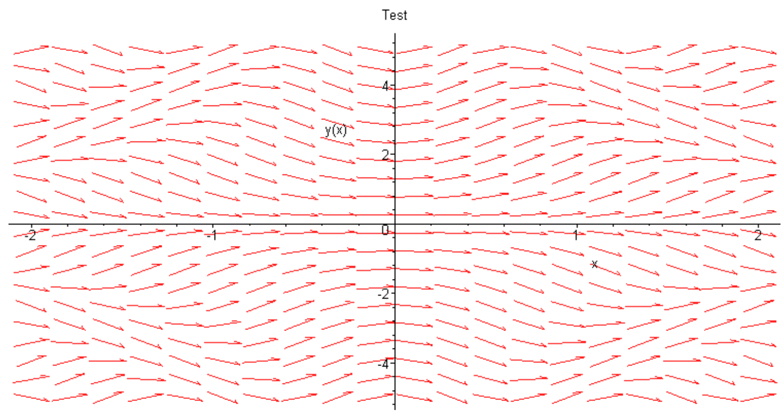
b) $\frac{dy}{dx} = 3x - y$

c) $\frac{dy}{dx} = -\frac{x}{y}$

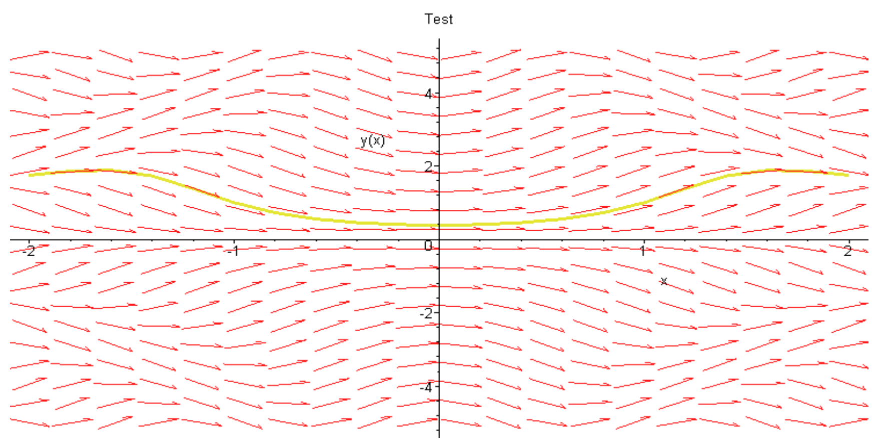
Ex: Sketch the direction field for the following differential equation.

$$\frac{dy}{dx} = 2\sin(xy)$$

> **dfieldplot(diff(y(x),x)=2*sin(x*y(x)),y(x),x=-2..2,y=-5..5,title='Test');**



> **DEplot(diff(y(x),x)=2*sin(x*y(x)),y(x),x=-2..2,[[y(1)=1.0]],y=-5..5,title='Test');**



Autonomous Equations and Equilibrium Solutions:

What is an Autonomous Process?

Def: An autonomous differential equation is one for which the derivatives of the dependent variable y depend only on the current value of y. More precisely, $\frac{dy}{dx} = f(y)$.

Def: Let $y' = f(y)$ be an autonomous first-order DE. An equilibrium solution is a constant function $y(x) = c$, such that $f(c) = 0$

The graph of $f(y)$ vs y ➔ Phase – diagram.

Ex: Sketch phase – diagram, isocline and then determine stable/unstable or semi-stable equilibrium solutions

These are autonomous DE

a) $$\frac{dy}{dx} = -2y^3 + 9y^2 - 9y$$

b) $$\frac{dy}{dx} = \sin(2y - \pi)$$

Equilibrium solutions are often of interest, because they are easy to analyze qualitatively and because often provide important information about the behavior of a system. For example, suppose that a population of birds has been constant for a long time (an equilibrium state) but pesticide contamination kills some of the birds. If the contamination is cleaned up, will the population grow back to its previous level?

This kind of question concerns the stability of the equilibrium state. Suppose a system is initially at equilibrium and is perturbed in some way. (A perturbation is a disturbance.) Will the system return to the equilibrium? If the answer is yes, then we say that the equilibrium is stable. (This is one of many notions of stability in mathematics.)

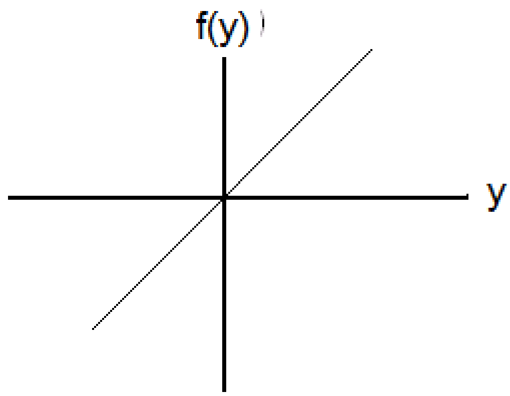
Def: Let $y = y_0$ be an equilibrium solution of the differential equation $y' = f(y)$. The equilibrium solution is stable if solution curves starting at initial values near y_0 return to y_0 as $t \to \infty$. Otherwise, the equilibrium is unstable.

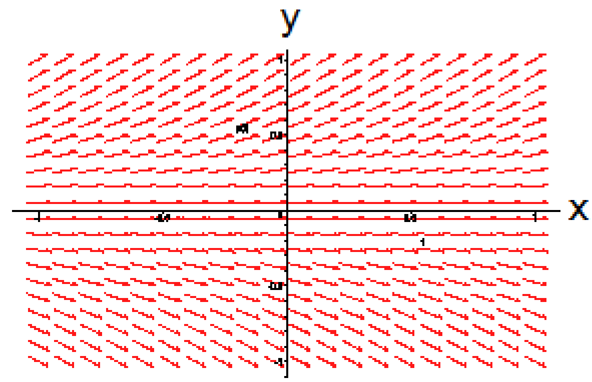
Ex: The population of some species of plants and animals does not recover if the number of individuals in a given area drops below a certain threshold. A simple model of a population of a certain species is given by

$$\frac{dp}{dt} = p(p-1)(2-p)$$

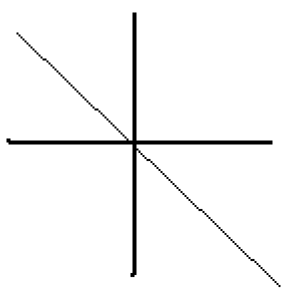
Phase Diagrams:

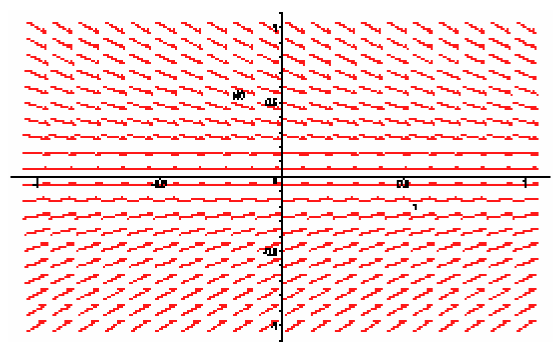
All of the information that is needed to analyze the qualitative behavior of the solution can be extracted from a plot of $f(y)$ verse y. Such a plot is called the phase diagrams:





- The phase diagram does not contain explicit information about the independent variable t.
- The slope lines in the direction field point up for those values of p for which $f(p) > 0$ and point down when $f(p) < 0$
- The constant function $p(t) = 0$ is an equilibrium solution. The existence of an equilibrium solution is apparent from the phase diagram, when the graph of $f(p)$ crosses the p axis at p = 0.

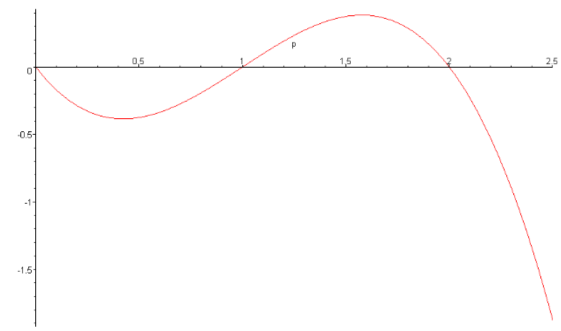


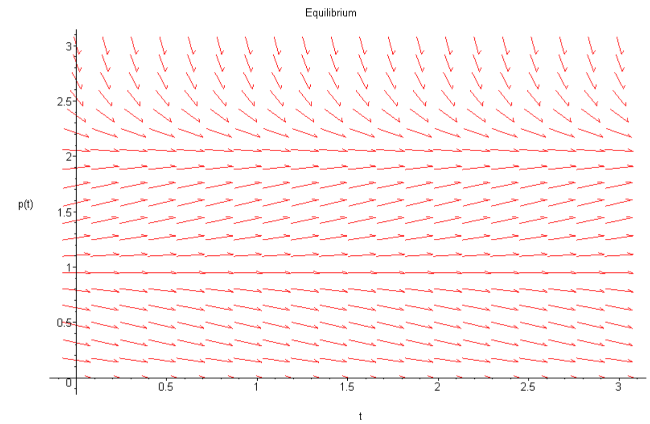


Theorem: Let $p' = f(p)$ for some differential function f, and suppose that $p(t) = p_0$ is an equilibrium solution.

If $\left.\frac{df}{dp}\right|_{p=p_0} > 0 \Rightarrow p = p_0$ is unstable. If $\left.\frac{df}{dp}\right|_{p=p_0} < 0 \Rightarrow p = p_0$ is a stable equilibrium solution.

Ex: Let's look at the phase diagram of the previous example $\frac{dy}{dx} = f(y) = y(y-1)(2-y)$





$$\frac{df}{dy} = (y-1)(2-y) + y(2-y) - y(y-1)$$

$$\left.\frac{df}{dy}\right|_{y=0} = -2 \Rightarrow stable; \quad \left.\frac{df}{dy}\right|_{y=1} = 1 \Rightarrow unstable \quad \left.\frac{df}{dy}\right|_{y=2} = -2 \Rightarrow stable$$

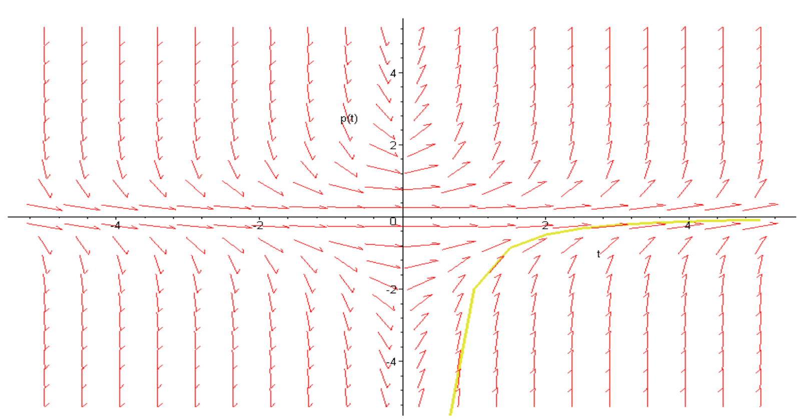
What if $\left.\frac{df}{dy}\right|_{y=y_0} = 0 \Rightarrow$ Then what?

Ex: $p' = f(p) = p^3 \Rightarrow \left.\frac{df}{dp}\right|_{p=0} = 3p^2\Big|_{p=0} = 0 \Rightarrow$ The theorem fails; in this case a direction field often is useful in this case.

Consider $p' = (p-1)^2; p' = -(p-1)^2 \; and \; p' = (p-1)^3$ Now, look at the phase diagram of each DE, then we can determine the whether the equilibrium solution p = 1 is stable, unstable or semi-stable.

<u>Non-Autonomous DE</u>

a) $p' = tp^2$



b) $p' = t + \sin p$