Section 7.2 *The Existence of the Laplace Transform and the Inverse Transform*

Def: A function f is called ***piecewise continuous*** on the interval [a,b] if we can divide [a,b] into a finite number of subintervals in such a manner that

1. f is continuous on each subinterval, and
2. f approaches a finite limit as the endpoints of each subinterval are approached from within.

Note: If f is piecewise continuous on every interval of the form [0,b], where b is a constant, then we say that f is piecewise continuous on $[0,\infty)$

Def: A function f is said to be of exponential order if there exist constants M and α such that $|f(t)| < Me^{\alpha t}$ for all $t > 0$

Ex: $f(t) = e^{5t}\sin 2t$, then clearly $|f(t)| = |e^{5t}\sin 2t| \le e^{5t}$ for all t, here we choose $M = 1; \alpha = 5$

However, if we look at $f(t) = e^{t^2}$ is NOT exponential order?

Because: $\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t(t-\alpha)} = \infty$ for any α. Consequently, e^{t^2} grows faster than $e^{\alpha t}$ for every choice of α

Lemma: (The comparison test for improper integrals):
Suppose that $0 \le G(t) \le H(t)\ for\ t \in [0,\infty)$.
If $\int_0^\infty H(t)dt$ converges, then so do does $\int_0^\infty G(t)dt$

Note: Let $E(0,\infty)$denote the set of all functions that are both **piecewise continuous** on $[0,\infty)$ and of **exponential order**. Which is a subspace of the vector space of all functions defined on $[0,\infty)$.

Theorem: If $f(x) \in E(0,\infty)$, then there exists a constant α such that $L[f] = \int_0^\infty e^{-st} f(t)dt$ exists for all $s > \alpha$.

Proof: Since f is piecewise continuous on $[0,\infty)$, $e^{-st} f(t)$ is integrable over any finite interval. Further, since $f(x) \in E(0,\infty)$, there exist constants M and α such that $|f(t)| \le Me^{\alpha t}$ for all $t > 0$. We now use the comparison test for integrals to establish that the improper integral defining the Laplace transform converges. Let

$$F(t) = |e^{-st} f(t)| \text{ Then } F(t) = e^{-st}|f(t)| \le Me^{-st}e^{\alpha t} = Me^{(\alpha - s)t}$$

But for $s > \alpha \Rightarrow \int_0^\infty Me^{(\alpha-s)t}dt = \lim_{N\to\infty}\int_0^N Me^{(\alpha-s)t}dt = \dfrac{M}{s-\alpha}$

Applying the comparison test for improper integrals with $F(t)$ as just defined as $G(t) = e^{(\alpha-s)t}$, it follows that $\int_0^\infty |e^{st} f(t)|dt$ converges for $s > \alpha$ and hence $\int_0^\infty e^{-st} f(t)dt$ converse as well. Thus, we have shown that $L[f]$ exists for $s > \alpha$, as required.

The inverse Laplace transform:

Let V denote the subspace of $E(0,\infty)$ consisting of all continuous functions of exponential order. We have seen in the previous section that the Laplace transform satisfies

$$L[f+g] = L[f] + L[g] \text{ and } L[cf] = cL[f].$$

Consequently, L defines a linear transformation of V onto Rng(L). Further, it can be shown that L is also one-to-one, and therefore, it has inverse transformation L^{-1}.

Def: The linear transformation $L^{-1} : Rng(F) \to V$ denoted by $L^{-1}[F](t) = f(t) \Leftrightarrow L[f](s) = F(s)$ is called the Inverse Laplace Transforms.

Note: We emphasize the fact that L^{-1} is a linear transformation, so that

$L^{-1}(F+G) = L^{-1}(F) + L^{-1}(G)$ and $\qquad L^{-1}(cF) = cL^{-1}(F)$

So far, we have:

$f(t) = 1 \Rightarrow$

$f(t) = t^n \Rightarrow$

$f(t) = e^{at} \Rightarrow$

$f(t) = \sin(bt) \Rightarrow$

$f(t) = \cos(bt) \Rightarrow$

$f(t) = \sinh(bt) \Rightarrow$

$f(t) = \cosh(bt) \Rightarrow$

Ex: Find the inverse Laplace Transform $L^{-1}[F](t)$ of the following:

a) $F(s) = \frac{3}{s^3} \Rightarrow$

b) $F(s) = \frac{2}{s+4}$

c) $F(s) = \frac{-2s+6}{s^2+4}$

d) $F(s) = \frac{3s+2}{s^2-3s+2}$

e) $$F(s)=\frac{7s^3+s^2+5}{3s^4+5s^2}$$

f) $$F(s)=\frac{4s^2+7s-62}{(s-4)(s+2)(s-3)}$$

g) $$F(s)=\frac{13s^2-8s-5}{(s^2+3)(3s-1)}$$

h) $F(s)=\dfrac{8-3s-5s^2}{(2s+1)(3s^2+2)}$

Theorem: Suppose that f is of exponential order on $[0,\infty)$ and that f' exists and is piecewise continuous on $[0,\infty)$. Then $L[f']$ exists and is given by $L[f']=sL[f]-f(0)$

Proof:

Theorem: If $f, f', ..., f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}(x)$ is piecewise continuous on $[0,\infty)$, then

$$L\left\{f^{(n)}(x)\right\} = s^n L(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

Solving Linear ODEs: It is apparent from the general result given above that $L\left(\frac{d^n y}{dt^n}\right)$ depends on $F(s) = L(y(t))$ and the n-1 derivatives of y(t) evaluated at t = 0. So, for the IVP which the DE has constant coefficients are ideally suited for Laplace. We have this $a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_0 y = g(x)$ with $y(0) = y_0, y'(0) = y_1, ..., y^{(n-1)}(0) = y_{n-1}$ where $a_i = constant$ for all i. By linearity property of the Laplace transform of this linear combination is a liner combination of $a_n L\left(\frac{d^n y}{dx^n}\right) + a_{n-1}L\left(\frac{d^{n-1}y}{dx^{n-1}}\right) + ... + a_0 L(y) = L(g(x))$

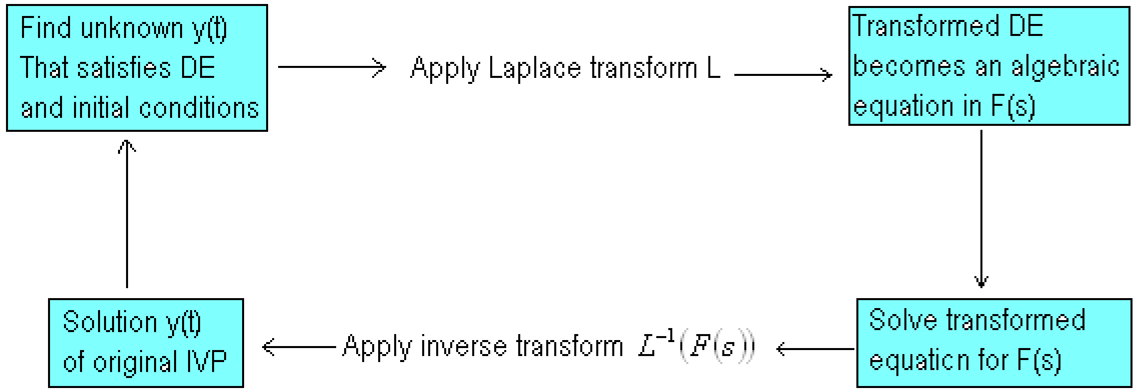
So, we have:

$$a_n\left[s^n F(s) - s^{n-1}y(0) - ... - y^{n-1}(0)\right] + a_{n-1}\left[s^{n-1}F(s) - s^{n-2}y(0) - ... - y^{(n-2)}(0)\right] + ... + a_0 F(s) = G(s)$$

Where $L(y(t)) = F(s)$ and $L(g(t)) = G(s)$.

In other words, the Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $F(s)$. If we solve the general transformed equation for the symbol $Y(s)$, we first obtain

$P(s)F(s) = Q(s) + G(s) \Rightarrow F(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$ *where* $P(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_0$ and $Q(s)$ is a polynomial in s of degree less than or equal to n-1 consisting of the various products of the coefficients a_i, $i = 1,2,...,n$ and the prescribed initial conditions $y_0, y_1, ..., y_{n-1}$ and $G(s)$ is the Laplace transform of $g(t)$. Finally, the solution $y(t) = L^{-1}(F(s))$



Ex: Use the Laplace transform to solve the initial-value problem

a) $\frac{dy}{dt} + 3y = 13\sin(2t),\ y(0) = 6$

b) $y''-3y'+2y=e^{-4t}, y(0)=1, y'(0)=5$

c) $y''-3y'+2y=3\cos t+\sin t;\ y(0)=1; y'(0)=1$