Preliminary Theory --- nth order Linear DE

Initial – Value and Boundary – value problems

Let $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)=g(x)$

Subject to $y(x_0)=y_0; y'(x_0)=y_1;...; y^{(n-1)}(x_0)=y_{n-1}$

Existence of a unique solution: Let $a_n(x),\ a_{n-1}(x),...,a_1(x),a_0(x)\ and\ g(x)$ be continuous on an interval I and let $a_n(x)\neq 0$ for every x in I. If $x=x_0\in I$, then a solution $y(x)$ of the initial value problem exists on the interval I and is unique.

Ex: Determine an interval I where the DE has a unique solution:

a) $\ln(x-1)y'''+\sin(x)y''-7y'+5y=e^{2x};\ y(3)=2, y'(3)=0, y''(3)=4$

b) $e^x y''-\frac{1}{x}y'+5y=x^2;\ y(0)=2, y'(0)=-1$ (not continuous at x =0)

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c) $(x^2-4)y''+3y'+\cos(x)y=e^x;\ y(0)=1, y'(0)=3$

Boundary – value problems: initial value at different points:

Ex: $a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)\ \ subject\ to\ y(a)=y_0;\ y(b)=y_1$

That means a solution is a function satisfying the DE on some interval I, containing a and b, whose graph passes through the two points $(a,y_0)\ and\ (b,y_1)$

For second order boundary – value problems: the initial values could be

$y'(a)=y_0;\ y(b)=y_1\ \ or\ y(a)=y_0;\ y'(b)=y_1\ \ or\ \ y'(a)=y_0;\ y'(b)=y_1$

Ex: Suppose that $y=A\cos(4x)+B\sin(4x)$ be a solution to $\frac{d^2y}{dx^2}+16y=0;$

a) $y(0)=0;\ y\left(\frac{\pi}{2}\right)=0;$

b) $y(0)=3;\ y\left(\frac{\pi}{3}\right)=0$

c) $y(0)=0;\ y\left(\frac{\pi}{2}\right)=1$

Homogeneous DE: $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)=0 \Rightarrow L(D)y=0$

Non-Homogeneous DE: $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)=g(x)\neq 0 \Rightarrow L(D)y=g(x)$

For now, we assume that $a_n(x)+a_{n-1}(x),\ldots a_1(x),a_0(x),g(x)$ are all continuous and $a_n(x)\neq 0$ for all x in I.

<u>Theorem</u>: Superposition Principle – Homogenous DE

Let $y_1(x),y_2(x),\ldots,y_k(x)$ be solutions of an nth – order homogenous DE $L(y_i)=0\ for\ i=1,2,\ldots,k$ over an interval I. Then the linear combination $y(x)=c_1y_1(x)+c_2y_2(x)+\ldots+c_ky_k(x)$ for constants $c_i;\ i=1,2,3,\ldots,k$ is also a solution over the same interval I.

Corollaries to theorem:

a) A constant multiple $y = c_1 y_1(x)$ is also a solution when $y_1(x)$ is a solution of $L(y) = 0$

b) A homogenous DE, $L(y) = 0$ always possesses the trivial solution $y = 0$

Ex: Let $y_1 = e^{-2x}$ *and* $y_2 = e^{5x}$ both are solutions for $y'' - 3y' - 10 = 0$ on interval R. Verify that $y = c_1 e^{-2x} + c_2 e^{5x}$ is also a solution.

Let y(x) be a solution of DE➔ We say solutions generated by y(x) are all solutions of the form $\alpha y(x)$

For Linear Dependence (LD) / Linear Indepdence (LI)

Def: A set of function $\{y_1, y_2, ..., y_n\}$ is said to be linearly dependent on an interval I, if there exist constants $c_1, c_2, ..., c_n$ not all zero, such that $c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0$ for every x in the interval I. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

Ex: The set of functions $\{\sin^2 x, \cos^2 x, \sec^2 x, \tan^2 x\}$ is linearly dependent over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$c_1 \sin^2 x + c_2 \cos^2 x + c_3 \tan^2 x + c_4 \sec^2 x = 0$$

$$choose\ c_1 = c_2 = 1;\ c_3 = 1; c_4 = -1$$

Def: Wronskian: Suppose $\{f_1(x), f_2(x), ..., f_n(x)\}$ possesses at least n – 1 derivative. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ...f_n \\ f_1' & f_2' & ...f'_n \\ ...f_1^{(n-1)} & f_2^{(n-1)} & ...f_n^{(n-1)} \end{vmatrix}$$

Theorem: if $W(f_1, f_2, ..., f_n) \neq 0$ for some x in I, then $\{f_1(x), f_2(x), ..., f_n(x)\}$ is linearly independent over I.

Ex: Determine whether the following functions are linearly independent / linearly dependent.

a) $\{e^x, x^2 e^x\}$

b) $\{x, x+x^2, 2x-x^2\}$

Def: Fundamental Set of Solutions:

Any set $\{f_1(x), f_2(x), ..., f_n(x)\}$ of n linearly independent solution of the homogeneous linear nth – order DE on an interval I is said to be a fundamental set of solution.

Theorem: Existence of a Fundamental Set.

There exists a fundamental set of solutions for the homogeneous linear nth – order

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \ (L(y) = 0)$$

So, $\{y_1, y_2, ..., y_n\}$ is a fundamental set of solution to L(y)=0. Then

$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is a general solution of $Ly = 0$ for $c_1, c_2, ..., c_n$ be arbitrary constants.

Ex: Find a fundamental solutions to the following homogeneous DE:

a) $y'' - 2y' - 15y = 0$

b) $9y'' - 6y' + 8y = 0$

c) $(D^2 + 3)(D - 4)^3 y = 0$

Non-Homogeneous DE: $L(y)=g(x)$

To solve $L(y)=g(x)$ ➔ first solve the homogeneous DE, $L(y)=0$ by finding its fundamental solutions➔ $y_h=c_1y_1+c_2y_2+\ldots+c_ny_n$ where $L(y_i)=0 \;\; for \;\; i=1,2,\ldots,n$. Then find y_p be a solution for the non-homogeneous DE: i.e. $L(y_p)=g(x)$ (particular solution, free of multiple of a constant)

Then the general solution to $L(y)=g(x)$ is $y(x)=y_h+y_p$

Ex: Suppose a particular solution $y_p=-2\cos(3x)$ for the DE: $y''+5y=8\cos(3x)$

$y''+5y=8\cos(3x)$; $y_p=-2\cos(3x)$; $y_c=c_1\cos\left(\sqrt{5}x\right)+c_2\sin\left(\sqrt{5}x\right)$

General solutions: $y(x)=c_1\cos\left(\sqrt{5}x\right)+c_2\sin\left(\sqrt{5}x\right)-2\cos(3x)$

Ex: a) Verify that $y_p=14\cos(3x)+2\sin(3x)$ is a solution of $y''-y'-12y=-300\cos(3x)$

b) Find a general solution of $y''-y'-12y=-300\cos(3x)$

Theorem: Superposition Principle for Non-homogeneous DE:

Given $L(D)y = g_1(x) + g_2(x) + \ldots + g_n(x)$

With particular solutions: $y_{p_1}, y_{p_2}, \ldots, y_{p_n}$ *for* $L(D)y_{p_i} = g_i(x)$ *for* $i = 1, 2, \ldots, n$.

Then particular solution: $y_p = y_{p_1} + y_{p_2} + \ldots + y_{p_n}$ is a solution for $L(D)y = g_1(x) + g_2(x) + \ldots + g_n(x)$

Ex: a) Verify that $y_{p_1} = 5x^2 - 3x + 2$ is a particular solution for $y'' + 2y' - 15y = -75x^2 + 65x - 26$

b) Verify that $y_{p_2} = 4xe^{-5x}$ is a particular solution for $y'' + 2y' - 15y = -8e^{-5x}$

c) Find a general solution for $y'' + 2y' - 15y = -75x^2 + 65x - 26 - 8xe^{-5x}$