Chapter 6 ***Series solutions of Linear DE***

6.1 Review power series:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_{n3}(x-c)^3 + \ldots$$

Power series of representation of basic functions:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ ; \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n; \ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n; \ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}; \ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note: For each power series, there exists an interval of x where the power series is convergence:

Def: A function f is said to be analytic at a point x = c if it can be represented by a power series in $x-c$ with either a positive or an infinite radius of convergence.

How to manipulate power series: Shifting the series index.

Ex: $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}$

Ex: Assume that the coefficients in the expansion of $f\left(x\right)=\sum_{n=1}^{\infty}a_n x^n$, satisfy the equation

$$\sum_{n=1}^{\infty}na_n x^n-\sum_{k=0}^{\infty}a_n x^{n+1}=0$$

6.2 Solutions About Ordinary Points

Given a DE:
$$a_2(x)y''+a_1(x)y'+a_0(x)y=0;\ a_2(x)\neq 0$$
$$y''+P(x)y'+Q(x)y=0;\ \text{Standard form}$$

Def: A point $x=x_0$ is said to be an ordinary point of a DE in standard form where P(x) and Q(x) are analytic at $x=x_0$. A point that is not an ordinary point of the DE, is said to be a singular point of the DE.

Ex: Solve $y''-2xy'-4y=0$ by the power series: $y=\sum_{n=0}^{\infty}a_nx^n$

Ex: Solve $y''+x^2y'-3xy=0$ by $y=\sum_{n=0}^{\infty}a_nx^n$ up to x^5

Sol:

Ex: Determine the term up to x^5 of the solution $y=\sum_{n=0}^{\infty} a_n x^n$ for $y''+\left(2-4x^2\right)y'-8xy=0$

Sol: