Chapter Eight
Systems of Differential Equations
First – Order Linear Systems

Def:

$$\frac{dx_1}{dt} = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + ... + a_{1n}(t)x_n(t) + F_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + ... + a_{1n}(t)x_n(t) + F_2(t)$$
$$...$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1(t) + a_{12}(t)x_2(t) + ... + a_{1n}(t)x_n(t) + F_n(t)$$

We can rewrite it as vectors: $\overrightarrow{x'(t)} = A(t)\overrightarrow{x(t)} + \overrightarrow{F(t)}$, where

$$\overrightarrow{x'(t)} = A(t)\overrightarrow{x(t)} + \overrightarrow{F(t)}$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ ... \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & ... & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & ... & a_{2n}(t) \\ ... & ... & ... & ... \\ a_{n1}(t) & a_{n2}(t) & ... & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ ... \\ x_3(t) \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \\ ... \\ F_n(t) \end{bmatrix}$$

Solution: of a system $\begin{bmatrix} x_1(t) \\ x_2(t) \\ ... \\ x_n(t) \end{bmatrix}$

Ex: $\begin{cases} \frac{dx}{dt} = 5x + 3y + e^{2t} \\ \frac{dy}{dt} = x - 2y + 5\cos(2t) \end{cases}$ => $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 5\cos(2t) \end{bmatrix}$

Theorem: Let the entries of the matrices A(t) ad F(t) be functions continuous on a common interval I that contains the point t_0. Then there exists a unique solution of the initial – value on the interval.

Theorem: Let $\overrightarrow{x_1(t)}, \overrightarrow{x_2(t)}, \overrightarrow{x_3(t)}, ..., \overrightarrow{x_n(t)}$ be a set of a solution vectors of the homogenous system on an interval I. Then the linear combination

$$\overrightarrow{x(t)} = c_1\overrightarrow{x_1(t)} + c_2\overrightarrow{x_2(t)} + c_3\overrightarrow{x_3(t)} + ... + c_n\overrightarrow{x_n(t)}$$

for $c_1, c_2, ..., c_n$ are arbitrary constants is also a solution of the system on the interval I.

Definition:

Let $\overrightarrow{x_1(t)}, \overrightarrow{x_2(t)}, \overrightarrow{x_3(t)}, ..., \overrightarrow{x_n(t)}$ be vectors in $V_n(I)$. Then the Wronskian of these vectors function denoted: $W[x_1, x_2, ..., x_n] = \det([x_1, x_2, ..., x_n])$

Note: The Wronskian of these vectors is defined as column vectors function in $V_n(I)$, whereas the Wronskian previously introduced refers to functions in $C^n(I)$

Theorem: The set $\overrightarrow{x_1(t)}, \overrightarrow{x_2(t)}, \overrightarrow{x_3(t)}, ..., \overrightarrow{x_n(t)}$ of vectors in $V_n(I)$ is linearly independent if the Wronskian is nonzero at some point in I.

Ex: Determine if $\overrightarrow{x_1(t)} = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}; \overrightarrow{x_2(t)} = \begin{bmatrix} 3\sin t \\ \cos t \end{bmatrix}$ is linearly independent.

$$W[x_1, x_2] = \det\begin{bmatrix} e^t & 3\sin t \\ 2e^t & \cos t \end{bmatrix} = e^t\cos t - 6e^t\sin t = e^t(\cos t - 6\sin t)$$

Theorem: Given $\frac{\overrightarrow{dx}}{dt} = A\overrightarrow{x(t)} + \overrightarrow{F(t)}$. The homogenous solution

$\overrightarrow{x_h(t)} = c_1\overrightarrow{x_1(t)} + c_2\overrightarrow{x_2(t)} + c_3\overrightarrow{x_3(t)} + ... + c_n\overrightarrow{x_n(t)}$ be solution of

$$\frac{\overrightarrow{dx}}{dt} = A\overrightarrow{x(t)} \Rightarrow \frac{\overrightarrow{dx_h}}{dt} = A\overrightarrow{x_h(t)}$$

And particular solution $\overrightarrow{x_p(t)} \Rightarrow \frac{\overrightarrow{dx_p}}{dt} = A\overrightarrow{x_p(t)} + \overrightarrow{F(t)}$.

Then the general solutions of the system are

$$\overrightarrow{x(t)} = \overrightarrow{x_h(t)} + \overrightarrow{x_p(t)} = \overrightarrow{x_h(t)} = c_1\overrightarrow{x_1(t)} + c_2\overrightarrow{x_2(t)} + c_3\overrightarrow{x_3(t)} + ... + c_n\overrightarrow{x_n(t)} + \overrightarrow{x_p(t)}$$

8.2 Homogeneous Linear Systems $\frac{\overrightarrow{dx}}{dt} = A_{n\times n}\overrightarrow{x}$

Def: Let A be an $n \times n$ matrix. Any values of λ for which $A\vec{v} = \lambda v$ has nontrivial solutions are called eigenvalues of A. The corresponding non-zero vectors v are called eigenvectors of A.

Ex: Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then we have that

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda v \Rightarrow \begin{cases} Eigenvalue\ \lambda = 3 \\ Eigenvector\ \overrightarrow{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda v \Rightarrow \begin{cases} Eigenvalue\ \lambda = -1 \\ Eigenvector\ \overrightarrow{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

How to find Eigenvalues and Eigenvectors of an n by n matrix A.

$A\overrightarrow{v} = \lambda\overrightarrow{v} \Rightarrow A\overrightarrow{v} - \lambda\overrightarrow{v} = \overrightarrow{0} \Rightarrow (A - \lambda I)\overrightarrow{v} = \overrightarrow{0}$

Procedure:

1. Find all scalars λ such that $\det(A - \lambda I) = 0$. These are the Eigenvalues.
2. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the distinct eigenvalues obtained from (1) then solve the k systems of linear equations $(A - \lambda_i I)v_i = 0;\ for\ \ i =, 1, 2, ..., n$ to find all eigenvectors v_i corresponds to each eigenvalue.

Note: The equation $p(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial.

Ex: Find all Eigenvalues and Eigenvectors of $A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}$

Ex: Find the Eigenvalues and Eigenvector of the following matrices

a) $A = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$

c) $A = \begin{bmatrix} -1/4 & 2 \\ -2 & -1/4 \end{bmatrix}$

Theorem: Let A be an $n \times n$ matrix with real elements. If λ is a complex eigenvalue of A with corresponding eigenvector v, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{v}$.

d) $A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$

Ex: Solve:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 2x + y \end{cases}; \quad \textit{verify} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is a solution:}$$

Homogeneous Constant Coefficient VDE: Non-Defective Coefficient Matrix

Theorem: Let $\frac{\overrightarrow{dx}}{dt} = A\overrightarrow{x}$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be n distinct real eigenvalues of the coefficient matrix A of the homogeneous system and let $\overrightarrow{v_1};\ \overrightarrow{v_2};\ \overrightarrow{v_3}; ..., \overrightarrow{v_n}$. Then the general solution of the system are $\overrightarrow{x(t)} = C_1 e^{\lambda_1 t}\overrightarrow{v_1} + C_2 e^{\lambda_2 t}\overrightarrow{v_2} + ... + C_n e^{\lambda_n t}\overrightarrow{v_n}$

We now consider the VDE: $\overrightarrow{x'(t)} = A\overrightarrow{x(t)}$

Ex: Solve the following:

a) $\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = -x_1 + 4x_2 \end{cases}$

b) $\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$

,

c) $\begin{cases} x_1' = 2x_1 + 2x_2 \\ x_2' = 2x_1 - x_2 \end{cases}$

d) $\overrightarrow{x'(t)} = A\overrightarrow{x(t)}$ where $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}$

Complex Eigenvalues:

Theorem: Let $u(t)$ *and* $v(t)$ be real-valued vector functions. If $w_1(t)=u(t)+iv(t)$ *and* $w_2(t)=u(t)-iv(t)$ are complex conjugate solutions to $x'=Ax$, then $x_1(t)=u(t)$ *and* $x_2(t)=v(t)$ are themselves real-valued solutions to $x'=Ax$

Theorem: Let A be a real 2 by 2 matrix with complex conjugate eigenvalues. Let $\lambda=a+bi$ be one of the eigenvalues, and let v be a corresponding eigenvector. The general solution of the system $x'=Ax$ is $x(t)=c_1\,\mathrm{Re}\left(e^{\lambda t}v\right)+c_2\,\mathrm{Im}\left(e^{\lambda t}v\right)$ where the constants c_1 *and* c_2 are real if the initial condition is real.

Ex: Solve $\begin{cases} x_1'=-x_1+2x_2 \\ x_2'=-2x_1-x_2 \end{cases}$ with initial value $\overrightarrow{x(0)}=\begin{bmatrix}1\\1\end{bmatrix}$

Sol: $\overrightarrow{x'(t)}=A\overrightarrow{x(t)}=\begin{bmatrix}x_1'(t)\\x_2'(t)\end{bmatrix}=\begin{bmatrix}-1 & 2\\-2 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$

By Euler's formula we have: $e^{(-1\pm 2i)} = e^{-t}\left(\cos 2t \pm i\sin 2t\right)$

Collect real and imaginary parts, and simplify to obtain: $\overrightarrow{x(t)} = e^{-t}\begin{bmatrix} \cos 2t + \sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$.

Ex: Find the general solution to the VDE $x' = Ax \text{ if } A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Ex: Find the general solution to the VDE $x' = Ax \text{ if } A = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}$

Vector Differential Equations: Defective Coefficient Matrix:

Double Real Eigenvalues:
If the characteristic equation of the 2 by 2 matrix A has a double root, then A has only one eigenvalue and typically only one eigenvector. Suppose p is an eigenvector corresponding to λ. The function $x(t) = e^{\lambda t} p$ is a solution because $x'(t) = \lambda e^{\lambda t} p = e^{\lambda t} Ap;\ because\ Ap = \lambda p$.
However, $x(t)$ is not a general solution.

Let A have an eigenvalue λ with only one corresponding eigenvector p. let q be a ***generalized eigenvector***, that is, a solution of the equation $(A - \lambda I)q = p$. Then the general solution is

$$x(t) = c_1 e^{\lambda t} p + c_2 e^{\lambda t}(q + tp)$$

Ex: Find the solution of the IVP $x' = Ax = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} x;\ x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$