Section 4.4: Constant Coefficient - Non-Homogeneous nth order DE
The method of Undetermined Coefficients: Annihilators

Def: Given $a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x) \Rightarrow L(D) = g(x)$

Note: To solve this DE, we solve as follows:

Step1: Solve the homogeneous DE: $L(D) = 0$. Namely, find the set of fundamental solutions, $\{y_1, y_2, ..., y_n\}$

Then the general solution for the homogeneous DE: $y_h = C_1 y_1 + C_2 y_2 + ... + C_n y_n$

i.e. $L(y_i) = 0 \;\; for \;\; i = 1, 2, ..., n$

Step 2: Solve the non-homogeneous DE ➔ particular solution: $L(D) = g(x)$ which means we need to find a function y_p, such that $L(y_p) = g(x)$.

Step 3: The general solution: $y(x) = y_h + y_p = C_1 y_1 + C_2 y_2 + ... + C_n y_n + y_p$

Ex: Solve: $y'' - y' - 20y = 40x^2 + 16x - 17; \quad y(0) = 1; \; y'(0) = -3$

Homogeneous solution:

$p(\lambda) = \lambda^2 - \lambda - 20 = 0 \Rightarrow (\lambda - 5)(\lambda + 4) = 0 \Rightarrow \lambda = 5, -4 \Rightarrow y_h = C_1 e^{5x} + C_2 e^{-4x}$

Given a particular solution: $y_p = 2x^2 + x - 1$

Then the general solution is:

How to find particular solution y_p such that $L(y_p) = g(x)$ ➔ We use method of undetermined coefficients for polynomial, exponential, sine and cosine functions.

1. $g(x) = polynomial$

Ex: Solve the following DE:

a) $y'' + 2y' - 15y = 30x^2 - 23x + 10$

b) $y'' - 6y' + 13y = 39x^2 + 29x - 115, \quad y(0) = 1;\ y'(0) = -2$

2. $g(x)$ is exponential functions.

c) $6y''+5y'-6y=220e^{4x}$

e) $y''' - 3y'' - y' + 3y = 15e^{-2x}$

f) $y'' - y' - 2y = 10\sin x;\ y(0) = 0;\ y'(0) = 1$

g) $6y'' - 7y' - 3y = 69\sin(2x) - 67\cos(2x),\ \ y(0) = 1,\ y'(0) = 2$

h) $y'' - y' - 20y = 8(1-4x)e^{4x}$

Duplicated root of homogeneous solutions

Ex: Solve the following:

a) $2y'' - 5y' - 3y = 14e^{3x}$

b) $y'' + 9y = 36\cos(3x) - 12\sin(3x)$

f) $3y''-5y'-2y=3e^{2x}+4e^{-x/3}$

g) $y''-4y'+29y=104\sin(5x)$ (Not duplicated homogeneous roots)

Multiple non-homogeneous DE:

Given $L(D)=g_1(x)+g_2(x)$ then solve $\begin{cases} L(D)=0 \text{ for homogeneous solution} \Rightarrow y_h \\ L(D)=g_1(x) \text{ for particular solution } g_1(x) \Rightarrow y_{p_1} \\ L(D)=g_2(x) \text{ for particular solution } g_2(x) \Rightarrow y_{p_2} \end{cases}$

Then the general solution of the DE is $y=y_h+y_{p_1}+y_{p_2}$

Ex: Solve the following DE:

a) $y''-2y'-3y=6x+7+6xe^{2x}$

Ex: Solve the following DE:

a) $6y''+7y'-3y=14e^{-2x}-6x^2+43x-23$

b) $y'' + 4y = 7\sin(2x) - 5\cos(2x) + 3e^{3x}$