Section 4.6 ***The Variation – of – Parameters method***

The method of undetermined coefficients have two severe limitations. Firstly, it is only applicable to DE with constant coefficients, and secondly, it can only be applied to DE whose nonhomogeneous terms are of the form described in section 2.4. For example, we could not use the method of undetermined coefficients to find a particular solution to the DE $y''+4y'-5y = x^2 \ln x$.

How to find a particular solution for this nonhomogeneous DE?

Theorem: Variation – of – Parameters.

Consider $y''+a_1 y'+a_2 y = F$ where $a_1, a_2,\ and\ F$ are assumed to be (at least) continuous on the interval I. Let $y_1\ and\ y_2$ be linearly independent solutions to the associated homogeneous equation $y''+a_1 y'+a_2 y = 0$ on I. Then a particular solution to the equation is $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. Where

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F \end{cases}$$

Then $u_1 = -\int_0^x \frac{y_2(t)F(t)}{W[y_1, y_2](t)}dt\ \ and\ \ u_2 = \int_0^x \frac{y_1(t)F(t)}{W[y_1, y_2](t)}dt$

Ex: Solve the following DE:

a) $y''+4y'+4y = e^{-2x}\ln x;\ x > 0$

b) $y'' + y = \tan x + 3x - 1 \quad for \; -\frac{\pi}{2} < x < \frac{\pi}{2}.$

c) $y''+9y = 6\cot^2 3x + 5e^{2x};\ 0 < x < \pi/6$

d) $4y'' + 36y = \csc 3x$

Variation of Parameter for nth order DE:

Consider the generalization of the variation-of-parameters method for linear nonhomogeneous DE of order n.

$$y^{(n)}+a_1(x)y^{(n-1)}+...+a_{n-1}(x)y'+a_n(x)y=F(x)$$

where assume that the function $a_1, a_2, ..., a_n,\ and\ F$ are at least continuous on the interval I. Let

$\{y_1(x), y_2(x), ..., y_n(x)\}$ be a LI set of solutions to the associated homogeneous equation

$$y^{(n)}+a_1(x)y^{(n-1)}+...+a_{n-1}(x)y'+a_n(x)y=0 \Rightarrow y_c(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$$

We now look for a particular solution to equation of the form

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)+...+u_n(x)y_n(x)$$

The idea is to substitute for y_p into the equation and choose the functions $u_1, u_2, ..., u_n$, so that the resulting y_p is indeed a solution.

Theorem: (Variation – of – Parameters)

Consider $y^{(n)}+a_1(x)y^{(n-1)}+...+a_{n-1}(x)y'+a_n(x)y=F(x)$ where assume that the function $a_1, a_2, ..., a_n,\ and\ F$ are at least continuous on the interval I. $\{y_1(x), y_2(x), ..., y_n(x)\}$ be a LI set of solutions to the associated homogeneous equation

$$y^{(n)}+a_1(x)y^{(n-1)}+...+a_{n-1}(x)y'+a_n(x)y=0 \Rightarrow y_c(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$$

We now look for a particular solution to equation of the form

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)+...+u_n(x)y_n(x)$$

Where $u_1, u_2, ..., u_n$ satisfies

$$\begin{aligned}
&y_1u_1'+y_2u_2'+...+y_nu_n'=0\\
&y_1'u_1'+y_2'u_2'+...+y_n'u_n'=0\\
&...\\
&y_1^{(n-1)}u_1'+y_2^{(n-1)}u_2'+...+y_n^{(n-1)}u_n'=F(x)
\end{aligned}$$

Ex: Find the general solution to $y'''-3y''+3y'-y=36e^x\ln x$.

Sol: The auxiliary polynomial of the associated homogeneous equation is $P(r)=r^3-3r^2+3r-1=(r-1)^3$. So that the solutions for homogeneous equation are $y_1=e^x; y_2=xe^x; and\ y_3=x^2e^x$. The particular solution is $y_p(x)=e^xu_1(x)+xe^xu_2(x)+x^2e^xu_3(x)$, where $u_1(x), u_2(x) and\ u_3(x)$ satisfies:

$$\begin{cases}
e^xu_1'+xe^xu_2'+x^2e^xu_3'=0\\
e^xu_1'+(e^x+xe^x)u_2'+(2xe^x+x^2e^x)u_3=0\\
e^xu_1'+(e^x+e^x+xe^x)u_2'+(2e^x+2xe^x+2xe^x+x^2e^x)u_3'=36e^x\ln x
\end{cases}$$

$$\Leftrightarrow \begin{cases}
u_1'+xu_2'+x^2u_3'=0\\
u_1'+(x+1)u_2'+(x^2+2x)u_3'=0\\
u_1'+(x+2)u_2'+(x^2+4x+2)u_3'=36\ln x
\end{cases}$$

$$\begin{bmatrix} 1 & x & x^2 & 0 \\ 1 & x+1 & x^2+2x & 0 \\ 1 & x+2 & x^2+4x+2 & 36\ln x \end{bmatrix} \approx \begin{bmatrix} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 2 & 4x+2 & 36\ln x \end{bmatrix} \approx$$

$$\begin{bmatrix} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 0 & 2 & 36\ln x \end{bmatrix} \Rightarrow 2u_3' = 36\ln x \Rightarrow u_3' = 18\ln x$$

$u_2' + 2xu_3' = 0 \Rightarrow u_2' = -2xu_3' = -2x(18\ln x) = -36x\ln x;$

$and\ u_1' + xu_2' + x^2u_3' = 0 \Rightarrow u_1' = -xu_2' - x^2u_3'$

$u_1' = -x(-36x\ln x) - x^2(18\ln x) = 18x^2\ln x$

$u_1(x) = 18\int x^2 \ln x dx = 2x^3(3\ln x - 1)$

$u_2(x) = -36\int x\ln x dx = 9x^2(1 - 2\ln x)$

$u_3(x) = 18\int \ln x dx = 18x(\ln x - 1)$

$y_p(x) = 2x^3e^x(3\ln x - 1) + xe^x(9x^2(1-2\ln x)) + x^2e^x(18x(\ln x - 1)) = \ldots = x^3e^x(6\ln x - 11)$

$\Rightarrow$ The general solution to the given DE is therefore

$y(x) = e^x\left[c_1 + c_2x + c_3x^2 + x^3(6\ln x - 11)\right]$