Nth – order –Constant Coefficient - linear DE.

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)$ wher $a_n, a_{n-1}, \ldots, a_1, a_0$ are constant.

We define: Differential Operator means: $D = y';\ D^2 = y'';\ldots;D^n = y^{(n)}$

So, $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \Rightarrow \underbrace{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y}_{L(D)} = g(x)$

So, for $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)$ is equivalent to $L(D) = g(x)$

Note that the DE: $\begin{cases} L(D) = 0 \Leftrightarrow Homogeneous \\ L(D) = g(x) \Leftrightarrow Non-Homogeneous \end{cases}$

Simplest case: nth – order – constant – coefficient – Homogeneous DE.

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \Rightarrow L(D) = 0 \; where \; a_n, a_{n-1}, \ldots, a_1, a_0$ are constants.

Find characteristic polynomial: $P(\lambda) = L(\lambda)$; then find the root of $P(\lambda) = 0 \;\; for \; \lambda$

<u>Case1</u>: $P(\lambda) = 0$ where $\lambda = \lambda_1; \lambda_2; \ldots; \lambda_n$ are distinct real solutions, namely: $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$

Then the solutions are: $y_1 = e^{\lambda_1 x};\ y_2 = e^{\lambda_2 x}; y_3 = e^{\lambda_3 x};\ldots; y_n = e^{\lambda_n x}$: These are linearly independent set of functions which called a “fundamental set of solutions of DE”. Put all these functions in the following fashion, we call a “general solution of a DE” $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + \ldots + C_n e^{\lambda_n x}$

Ex: Solve the following DE:

a) $2y''' + y'' - 18y' - 9y = 0$

b) $(3D-2)(D+3)(D-4)(D+5)y=0$

IVP: (Initial Value Problems) for n^th – order – Homogeneous - DE:

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \Rightarrow L(D)y = 0$

Subject to n conditions: $y(x_0) = y_0; y'(x_0) = y_1;\ y''(x_0) = y_2; \ldots; y^{(n-1)}(x_0) = y_{n-1}$

Ex: Solve the following IVP:

a) $y''' - y'' - 20y' = 0;\ y(0) = 3;\ y'(0) = 12;\ y''(0) = -3$

b) $6y'' - y' - 2y = 0;\ y(0) = 3;\ \ y'(0) = 1$

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c) $y''' + 2y'' - 5y' - 6y = 0;\ y(0) = y'(0) = 0;\ y''(0) = 1$

Case 2: Solve $L(D)=0 \Rightarrow P(\lambda)=0$ where λ are repeated real solutions. Suppose $\lambda = \underbrace{\lambda_1 = \ldots = \lambda_1}_{n\ times}$

Then the solution: $y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_2 x} + C_3 x^2 e^{\lambda_3 x} \ldots + C_n x^{n-1} e^{\lambda_n x}$

Ex: Solve:

a) $D^2(D+4)^3(D+3)^2\, y = 0$.

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b) $y''' + 3y'' - 4y = 0$

c) $2y''' + y'' - 4y' - 3y = 0;\ \ y(0) = 1; y'(0) = 0;\ y''(0) = 1$

<u>Case 3</u>: Solve $L(D) = 0$ where $P(\lambda) = 0$ and $\lambda = \alpha \pm \beta i$

Then the solution: $y = C_1 e^{(\alpha+\beta i)x} + C_2 e^{(\alpha-\beta i)x} = e^{\alpha x}\left(C_1 e^{\beta i x} + C_2 e^{-\beta i x}\right)$

Euler's formula: $e^{\theta i} = \cos\theta + i\sin\theta$

Ex: Solve the following DE:

a) $4y'' + 12y' + 13y = 0$

b) $\left(D+2\right)^3\left(D^2+10D+29\right)y=0$

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c) $\left(D^2+6D+12\right)^2\left(D^2+5\right)^2 y=0$

Ex: Solve the following IVP:

a) $y''-4y'+5y=0;\ y(0)=2; y'(0)=7$

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b) $y'''+3y''+5y'+15y=0;\ y(0)=0;\ y'(0)=1;\ y''(0)=-1;$