Section 2.5 Change of Variables

Def: A function $f(x,y)$ is said to be homogeneous of degree zero if $f(tx,ty)=f(x,y)$ for all positive values of t for which (tx,ty) is in the domain of f.

Ex: Verify if the following functions are homogeneous of degree zero

$a)\quad f(x,y)=\dfrac{x}{y};$

$b)\quad f(x,y)=\dfrac{\sqrt{3x^2+xy+5y^2}}{7x-3y}$

Theorem: A function $f(x,y)$ is homogeneous of degree zero if and only if it depends on y/x only.

Proof: Suppose that f is homogenous of degree zero. We must consider two cases separately.

(a) if x>0, we can take $t=1/x \Rightarrow f(x,y)=f(1/t,y)=f(1,yt)=f(1,y/x)$ which is a function of y/x.

(b) if x<0, then we can take $t=-1/x \Rightarrow f(x,y)=f(-1/t,y)=f(-1,-yt)=f(-1,-y/x)$, this clearly is also a function of y/x.

Conversely, suppose that $f(x,y)$ depends only on y/x. If replace x by tx and y by ty then f is unaltered and hence is homogeneous of degree zero.

Technique: If $f(x,y)$ is a homogenous of degree zero, then the DE, $\dfrac{dy}{dx}=f(x,y)$ is called a homogeneous first – order DE. So we see that $\dfrac{dy}{dx}=f(x,y)$, we can not solve it directly. However, we can think of $\dfrac{dy}{dx}=f(x,y)=F(y/x)$ for some function F and let y/x be a dummy variable we have $V=y/x \Rightarrow y=xV(x) \Rightarrow \dfrac{dy}{dx}=x\dfrac{dV}{dx}+V$, substituting into the DE, we have $x\dfrac{dV}{dx}+V=F(V)$ ➔ Separation of variables we have $x\dfrac{dV}{dx}=F(V)-V \Leftrightarrow \dfrac{dV}{F(V)-V}=\dfrac{dx}{x}$

Theorem: The change of variables $y=xV(x)$ reduces a homogeneous first-order DE $\dfrac{dy}{dx}=f(x,y)$ to the separable equation $\dfrac{dV}{F(V)-V}=\dfrac{dx}{x}$

Ex: Solve the following:

a) $$\frac{dy}{dx} = \frac{4x + y}{x - 4y}$$

b) $$\left(xy + y^2 + x^2\right)dx - x^2 dy = 0$$

b) $\left(x^2 + y^2\right)dx + \left(x^2 - xy\right)dy = 0$

Substitution and Transformations

When the right hand side of the equation $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of $ax + by$ where a and b are constants, that is $\frac{dy}{dx} = G(ax + by)$, then substitute $z = ax + by$

Ex: Solve the following:

a) $\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$

b) $$\frac{dy}{dx} = \sqrt{x+y} - 1$$

c) $$\frac{dy}{dx} = \cos(4x + 4y - 3)$$

Bernoulli Equations: A DE that can be written in the form $\frac{dy}{dx} + p(x)y = q(x)y^n$ where n is a constant.

Note: if $n \neq 0,1$, then a Bernoulli equation is non-linear, but can be reduced to a linear equation as follows:

Divide equation by y^n to obtain $y^{-n}\frac{dy}{dx} + y^{1-n}p(x) = q(x)$. Now let $u(x) = y^{1-n} \Rightarrow \frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Therefore, we have $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$. Substitute into the equation, we have

$\frac{1}{1-n}\frac{du}{dx} + u(x)p(x) = q(x) \Leftrightarrow \frac{du}{dx} + (1-n)u(x)p(x) = (1-n)q(x)$ ← This is a linear equation of u in term of x. We can solve for u of x.

Ex: Solve the following DE

a) $x\frac{dy}{dx} + y = x^2y^2$

b) $$\frac{dy}{dx}+\frac{3}{x}y=\frac{12y^{2/3}}{\left(1+x^2\right)^{1/2}};x>0$$

c) $$\frac{dy}{dx}-5y=-\frac{5}{2}xy^3$$

<u>Equations with Linear Coefficients</u>:

We have used various substitutions for y to transform the original equation into a new equation that we could solve. In some cases we must transform both x and y into new variables, say u and v. This is the situation for equations with linear coefficients, that is equation of the form

$$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0 \text{ Where } a_i's; b_i's \text{ and } c_i's \text{ are constants.}$$

When $a_1b_2 = a_2b_1$, then equation can be put into the form $\frac{dy}{dx}=G(ax+by)$.

Before considering the general case when $a_1b_2 \neq a_2b_1$, let's first look at the special situation when $c_1 = c_2 = 0$ then the equation becomes $(a_1x+b_1y)dx+(a_2x+b_2y)dy=0$, when can be rewritten in the form

$$\frac{dy}{dx}=-\frac{a_1x+b_1y}{a_2x+b_2y}=-\frac{a_1+b_1(y/x)}{a_2+b_2(y/x)} \Leftarrow \text{This is homogeneous of order zero, so let } V=\frac{y}{x}$$

Now, in general, when $a_1b_2 \neq a_2b_1$, then we seek a translation of axes of the form $x=u+h; y=v+k$ where h and k are constants, that will change $a_1x+b_1y+c_1$ into a_1u+b_1v and $a_2x+b_2y+c_2$ into a_2u+b_2v

$$a_1x+b_1y+c_1=a_1(u+h)+b_1(v+k)+c_1=a_1u+b_1v+a_1h+b_1k+c_1 \Rightarrow a_1h+b_1k+c_1=0$$

$$a_2x+b_2y+c_2=a_2(u+h)+b_2(v+k)+c_2=a_2u+b_2v+a_2h+b_2k+c_2 \Rightarrow a_2h+b_2k+c_2=0$$

So basically, we need to solve the system of linear equations $\begin{cases} a_1h+b_1k+c_1=0 \\ a_2h+b_2k+c_2=0 \end{cases}$ for h and k.

Then we have $\frac{dv}{du}=-\frac{a_1u+b_1v}{a_2u+b_2v} \Leftarrow$ Homogeneous of order zero.

<u>Ex</u>: Solve $(-3x+y+6)dx+(x+y+2)dy=0$