Section 8.4 Matrix Exponential Function.

Given a DE:

$$y' = ay \Rightarrow y = Ce^{at}$$

$$\overrightarrow{y'(t)} = A_{n\times n}\overrightarrow{y(t)} \Rightarrow \overrightarrow{y(t)} = \left[e^{At}\right]_{n\times n}\overrightarrow{C_{n\times 1}}$$

Taylor expansion (Maclaurin's series) $f(x) = e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Def: $e^{At} = I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{2!}t^3 + \ldots$

Properties of the matrix exponential function:

1. $e^{(A+B)t} = e^{At}e^{Bt}$
2. For all square matrices A, e^{At} is invertible and $\left(e^{At}\right)^{-1} = e^{-At} \Rightarrow e^{At}e^{-At} = I$

Ex: Compute e^{At} *if* $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Note: In general $If\ A = D\left(a_{11}, a_{22}, \ldots, a_{nn}\right) \Rightarrow e^{At} = D\left(e^{a_{11}t}, e^{a_{22}t}, \ldots, e^{a_{nn}t}\right)$

Theorem: Let A be a non-defective square matrix with n linearly independent eigenvectors: $B = \left\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\right\}$. Then $e^{At} = Pe^{Dt}P^{-1}$ where P is a matrix of eigenvectors and D is a diagonal matrix whose entries are eigenvalues of A.

Ex: Compute the function $e^{At}\ \ if\ \ A = \begin{bmatrix} 3 & 3 \\ 5 & 1 \end{bmatrix}$

How do we apply matrix exponential function to solve a system of DE:

Given $e^{At} = I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + ... = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{2!}t^3 + ...$

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\left(I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + ...\right) = A + A^2 t + \frac{A^3}{2!}t^2 + \frac{A^4}{3!}t^3 + ...$$

$$\frac{d}{dt}e^{At} = A\left[I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + ...\right] = Ae^{At} ==> \frac{\overrightarrow{dy}}{dt} = A\overrightarrow{y(t)}$$

If we let $\Phi = e^{At}$ $(fundamental\ \ matrix) \Rightarrow \Phi' = A\Phi\ \ and\ \ \Phi(0) = e^{A(0)} = I$ and

So, given a system of DE: $\frac{\overrightarrow{dx}}{dt} = A\overrightarrow{x(t)} + \overrightarrow{F(t)} \Rightarrow solution: \overrightarrow{x(t)} = e^{At}\vec{C} + e^{At}\int_{t_0}^{t} e^{-As}F(s)\,ds$

It's the same as section 8.3 ($\overrightarrow{x(t)} = \Phi(t)\vec{C} + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)F(s)\,ds$

Solve system of DE by Laplace Transform: $\frac{\overrightarrow{dX}}{dt} = A\overrightarrow{X};$ Let $X(t) = e^{At}$ be a solution

Then $X(0) = e^{A(0)} = I$

Apply Laplace both sides of the DE:

$$L\left(\frac{dX}{dt}\right) = L(AX)$$

$$sL(X) - X(0) = AL(X)$$

$$(sI - A)L(X) = I ==> L(X) = (sI - A)^{-1}$$

$$X(t) = L^{-1}\left((sI - A)^{-1}\right)$$

Ex: Using Laplace Transform to solve: $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$