Solving Ordinary Differential Equations by Maple:

**Step 1:** Initialize the program:
- with(DEtools):
- with(PDEtools):

**Step 2:** Define an ODE: (There are several ways of defining equations, we start with the basic one first, just basically give the ODE a name:

**First Order DE:**

Example: \( \frac{dy}{dx} - \frac{3}{x} y = \sin(x) \)

> with(DEtools):
> eq1 := diff(y(t),t) - (1/3)*y(t) = sin(t);

\[
\text{eq1} := \frac{d}{dt} y(t) - \frac{1}{3} y(t) = \sin(t)
\]

**Higher Order ODE:**

Example: \( m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0 \)

> eq2 := m*diff(y(t),t$2)+b*diff(y(t),t)+k*y(t)=0;

\[
\text{eq2} := m \frac{d^2}{dt^2} y(t) + b \frac{d}{dt} y(t) + k y(t) = 0
\]

Example:

\[
\frac{d^4 y}{dt^4} - 10 \frac{d^3 y}{dt^3} + 35 \frac{d^2 y}{dt^2} - 50 y + 24 = 5 e^t
\]

> eq3 := diff(y(t),t$4)-10*diff(y(t),t$3)+35*diff(y(t),t$2)-50*y(t)+24=5*exp(t);

**Solving ODE:**

> dsolve(eq1,y(t));

\[
y(t) = -\frac{9}{10} \cos(t) - \frac{3}{10} \sin(t) + e^{\frac{1}{3} t} \_C1
\]

> dsolve(eq2,y(t));

\[
y(t) = _C1 e^{\frac{b + \sqrt{b^2 - 4 k m}}{2 m} t} + _C2 e^{\frac{-b + \sqrt{b^2 - 4 k m}}{2 m} t}
\]

**Solving IVP**

Now, with equation #1, we want to solve it with initial value say \( y(0) = \pi \)

> dsolve({eq1,y(0)=Pi},y(t));

\[
y(t) = -\frac{9}{10} \cos(t) - \frac{3}{10} \sin(t) + e^{\frac{1}{3} t} \left( \pi + \frac{9}{10} \right)
\]
For equation #2, we want to solve with initial value \( y(0) = 10 , \ y'(0) = 2 \)

We can define the initial value as:
\[
IV := y(0) = 10 , \ D(y)(0) = 2
\]

and for equation #2, for instance, \( m =1, \ b = -2, \ k =0.1, \) we then substitute into equation #2 with new equation name:
\[
eq m =1, \ b = -2, \ k =0.1, \text{eq2};
\]

Now, we can solve eq2m with the initial value IV as follows:
\[
> \text{dsolve}({\text{eq2m}},y(t));
\]

1. **Now, use the outline above to solve the following ODE:**

   a) \( (1 + e^t) \frac{dy}{dt} + e^t y = 0; \ y(0) = 1 \)

   b) \( \frac{dy}{dx} + y = \sin x; \ y(1) = -\frac{\pi}{2} \)

   c) \( \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 7 y = 0; \ y(0) = 0; \ y'(0) = -1 \)

   d) \( \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 4 x = \cos 2 t; \ x(0) = \pi \)

   e) \( \frac{d^2 x}{dt^2} + 2 x = \sin (2 \sqrt{2} t); \ x(0) = 1; \ x'(0) = -1 \)

   f) \( y'''' - 3 y''' + 3 y'' - y = x - 4 e^x \)

   g) \( \frac{d^2 x}{dt^2} + \frac{dx}{dt} + y = x \sin x; \)

   h) \( 2 y'''' - 3 y''' - 3 y'' + 2 y = (e^x + e^{-x})^2 \)

**Application: Mixing Problems**

**Ex1**: A tank initially contains 40 gal of sugar water having a concentration of 3 lb of sugar for each gallon of water. At time zero, sugar water with a concentration of 4 lb of sugar per gal begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of the sugar-water solution in the tank remains constant.

(a) How much sugar is in the tank after 15 minutes?

(b) How long will it take the sugar content in the tank to reach 150 lb? 170 lb?

(c) What will be the eventual sugar content in the tank?

**Initialize**:

\[
> \text{restart};
> \text{with( DEtools )};
> \text{with( plots )};
> \text{with( PDEtools )};
\]

The mathematical formulation of this problem must express the physical requirement that
\[
\frac{d}{dt} (\text{amount of sugar in tank}) = (\text{rate sugar is added to tank}) - (\text{rate sugar is removed from tank})
\]

Let \( x(t) \) denote the amount of sugar (pounds) in the tank at time \( t \) (minutes). Then, the rates in and out are respectively.
> rate_in := 4 * 2;
> rate_out := (x(t)/40) * 2;

so that the governing ODE is

> ode := diff( x(t), t ) = rate_in - rate_out;

The amount of sugar in the tank initially, that is, when \( t = 0 \), gives the initial condition

> ic := x(0)=40 * 3;

The ODE in this IVP is first-order and linear. The integrating factor is

> mu(t) = intfactor( ode );

The solution to the IVP is

> sol := dsolve( { ode, ic }, x(t), [linear] );

(a) The amount of sugar (in pounds) in the tank after 15 minutes is

> eval( sol, t=15. );

(b) The tank will contain 150 pounds of sugar at a time \( t \) (in minutes) satisfying

> eq150 := eval( sol, x(t)=150 );

Thus, the desired time is found by the calculations

> t150 := solve( eq150, t );
> t[^`150 lbs`] = t150;
> `` = evalf(t150);

Repeating the same steps for the time when 170 pounds of sugar are in the tank leads to the equation

> eq170 := eval( sol, x(t)=170 );

whose solution is

> t170 := solve( eq170, t );
> t[^`170 lbs`] = t170;
> `` = evalf(t170);

This complex-valued solution is clearly not physically realistic. A quick inspection of the solution, graph with title “One tank mixing problem”

> plot( rhs(sol), t=0..120, title="One tank mixing problem" );

shows that the amount of sugar in the tank reaches a steady-state limit that is well below 170 pounds. Therefore, at no time is there ever 170 pounds of sugar in the tank.

(c) In (b) it was noted that the amount of sugar in the tank levels off below 170 pounds. The exact limit can be determined from the solution by looking at the limit as \( t \rightarrow \infty \), that is, at

> steady_state := map( Limit, sol, t=infinity );

whose value is

> value( steady_state );

Note that \( x = 160 \) is an equilibrium solution for this ODE. However, be careful to avoid the common error of concluding that the limit is 160 pounds because \( x = 160 \) is an equilibrium solution.

2. Use the outline above to solve the following:

   a) Initially, a 100 – liter tank contains a salt solution with concentration 0.5 kg/liter. A fresher solution with concentration 0.1 kg/liter flows into the tank at the rate of 4 liter/min. The contents of the tank are kept well stirred, and the mixture flows out at the same rate it flows in.
      i. Find the amount of salt in the tank as a function of time.
      ii. Determine the concentration of salt in the tank at any time.
      iii. Determine the steady – state amount of salt in the tank.
      iv. Find the steady – state concentration of salt in the tank.

   b) At the start, 5 lbs of salt are dissolved in 20 gal of water. Salt solution with concentration 2 lb/gal is added at a rate of 3gal/min, and the well – stirred mixture is drained out at the same rate of flow. How long should this process continue to raise the amount of salt in the tank to 25 lbs?
Consider the previous problem, except that the outflow from the tank is at a rate of 3 gallons per minute.

(a) Find the formula for the volume of sugar water in the tank at any time. When is the tank empty?
(b) Find the IVP for the amount of sugar in the tank.
(c) Find the IVP for the concentration of sugar in the water.
(d) When is the tank empty? What is the concentration of sugar immediately before the tank is empty? How much sugar is in the tank at this time?
(e) Plot the amount of sugar and concentration of sugar in the tank up to the time the tank becomes empty. What happens to these solutions at later times?

Solution:

a) The volume starts at 40 gallons. Every minute 2 gallons of sugar water are added to the tank and 3 gallons are removed; the net change is a loss of 1 gallon per minute. The rates "in" and "out" are respectively

\[
\begin{align*}
V_{\text{rate in}} & := 2; \\
V_{\text{rate out}} & := 3;
\end{align*}
\]

so the time-varying volume in the tank is

\[
V := 40 + (V_{\text{rate in}} - V_{\text{rate out}}) * t;
\]

Note that \( V(t) \) is the solution of the IVP

\[
\begin{align*}
\text{odeV} & := \text{diff}( v(t), t ) = 2 - 3; \\
v(0) & := v(0) = 40;
\end{align*}
\]

as confirmed via

\[
\text{dsolve( \{ odeV, icV \}, v(t), \{\text{separable}\} );}
\]

b) The IVP for the amount of sugar in the tank is similar to the one in the previous example. There is no difference in the rate at which sugar enters the tank. The concentration of sugar exiting the tank is

\[
\frac{x(t)}{V}
\]

and this is different because \( V = V(t) \) is no longer constant. The rates "in" and "out" are now respectively

\[
\begin{align*}
S_{\text{rate in}} & := rate\_\text{in}; \\
S_{\text{rate out}} & := (x(t)/V) * 3;
\end{align*}
\]

Thus, the governing ODE is

\[
\text{ode2} := \text{diff}( x(t), t ) = S_{\text{rate in}} - S_{\text{rate out}};
\]

The initial condition is unchanged:

\[
\text{ic2} := \text{ic};
\]

c) The IVP for the concentration of sugar in the tank is obtained from the ODE in (b) and the definition of the concentration, \( c(t) \), which is

\[
\text{conc_eq} := c(t) = x(t)/'V';
\]

Rather than deriving the differential equation for \( c(t) \) manually, the \text{dchange} command from the \text{PDEtools} package will be used to automate the process. It gives

\[
\text{odeC} := \text{dchange( x(t) = c(t)*V, ode2, [c] );}
\]

The initial condition for the concentration is

\[
\text{icC} := c(0) = 3;
\]

d) The tank is empty when the volume of sugar water is zero. This occurs after 40 minutes, as obtained by

\[
\text{t\_empty} := \text{solve( V=0, \{t\} );}
\]

The concentration is found by solving the (linear) IVP found in (c), which is done via

\[
\text{solC} := \text{dsolve( \{ odeC, icC \}, c(t), \{\text{linear}\} );}
\]

Thus, the concentration at the instant the tank empties is

\[
\text{subs( t\_empty, solC );}
\]

The amount of sugar in the tank as the tank empties is obtained by evaluating the solution

\[
\text{sol2} := \text{dsolve( \{ode2,ic2\}, x(t), \{\text{linear}\} );}
\]

at the time the tank empties. This gives

\[
\text{subs( t\_empty, sol2 );}
\]

which is exactly what one would expect. (If not, think about it!)

e) The requested plots are

\[
\text{plot( rhs(sol2), t=0..40, title='Amount of sugar');}
\]
The concentration remains positive until $t = 120$, but after $t = 40$, the volume and amount of sugar become negative. Even though the IVPs have solutions for all time, for $t > 40$ these results are not physically meaningful.

Now, follow the outline above to solve the following:

1. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing $\frac{1}{2}$ lb of salt per gallon is pumped into the tank at a rate of 3 gal/min. The well-mixed solution is then pumped out at a slower rate of 2 gal/min.
   i) Find the IVP for the amount of salt in the tank at time $t$.
   ii) Find the IVP for the concentration of sugar in the water at time $t$.
   iii) When will the tank overflow?
   iv) What will be the number of pounds of salt in the tank at the instant it overflows?
   v) Determine the number of pounds of salt in the tank as $t \to \infty$. Does your answer agree with your intuition?
   vi) Plot the graph of $x(t)$ on the interval $[0, 500]$.

2. Problems 4, 5, and 6 on page 65.