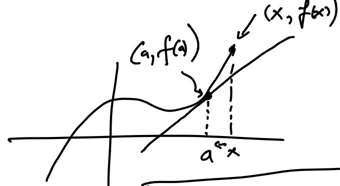
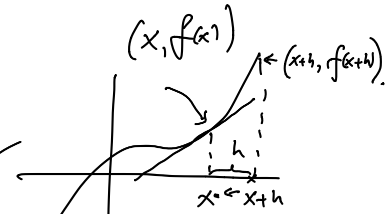
Review Math 180 materials:



Derivative:

1. Definition: Derivative of $f(x)$ at $\boxed{x=a}$

slope $m = \frac{\text{rise}}{\text{run}} = \frac{f(x) - f(a)}{x - a} \Rightarrow$ Derivative: $\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$ ✓



Derivative at any x: Derivative : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\cancel{x}+h - \cancel{x}} = \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$ ✓

2. Rules:

1. $(k)' = 0$
2. $(x^n)' = n x^{n-1}$
3. $(k \cdot f(x))' = k \cdot f'(x)$
4. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
5. $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + g'(x) \cdot f(x)$ $\left[(uv)' = u'v - v'u \right]$
6. $\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$ $\left[\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \right]$
7. $(f \circ g)' = f'(g(x)) \cdot g'(x)$. $\left\{ \begin{array}{l} (b^{u(x)})' = b^{u(x)} \cdot u'(x) \cdot \ln b. \\ (e^{u(x)})' = e^{u(x)} \cdot u'(x). \end{array} \right\}$
8. (Exponent)$'$: $(e^{kx})' = e^{kx} \cdot k$.
9. $(\log)'$: $(\ln x)' = \frac{1}{x}$ ✓ $\quad \left(\log_b (u(x))\right)' = \frac{1}{u(x)} \cdot u'(x) \cdot \frac{1}{\ln b}$.
10. All 6 trig functions: $\left\{ \begin{array}{l} (\sin x)' = \cos x \\ (\cos x)' = -\sin x \quad \ldots \end{array} \right.$
11. All 6 Inverse Trig. functions $\left\{ \begin{array}{l} (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \Rightarrow (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \\ (\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}} \Rightarrow (\csc^{-1} x)' = \frac{-1}{x\sqrt{x^2-1}} \\ (\tan^{-1} x)' = \frac{1}{1+x^2} \Rightarrow (\cot^{-1} x)' = \frac{-1}{1+x^2} \end{array} \right.$

3. Implicit Differentiation:

Variable x: $(x^n)' = n x^{n-1}$

function $y(x)$: $(y^n)' = n \cdot y^{n-1} \cdot y'$

Product Rule: $(\underbrace{x^m}_{u} \cdot \underbrace{y^n}_{v})' = \underbrace{m x^{m-1}}_{u'} \cdot \underbrace{y^n}_{v} + \underbrace{n y^{n-1} \cdot y'}_{v'} \cdot \underbrace{x^m}_{u}$.

Ex: Differentiate the following functions:

a) $$f(x) = \underbrace{3^{\sin(5x^2+1)}}_{u} \underbrace{\sqrt{\tan^{-1}(3x)+4x}}_{v}$$

$$f'(x) = \underbrace{3^{\sin(5x^2+1)} \cdot \cos(5x^2+1) \cdot 10x \cdot \ln 3}_{u'} \underbrace{\sqrt{\tan^{-1}(3x)+4x}}_{v} + A$$

$$A = \underbrace{\frac{1}{2}\left(\tan^{-1}(3x)+4x\right)^{-1/2}\left(\frac{3}{1+9x^2}+4\right)}_{v'} \cdot \underbrace{\boxed{3^{\sin(5x^2+1)}}}_{u}$$

b) $$f(x) = \cos^4\left(\frac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)$$

$$f'(x) = 4\cos^3\left(\frac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)\left[-\sin\left(\frac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)\right] \cdot A$$

$$A = \frac{(3x^2-10x)\sec^{-1}(3x+1) - \frac{3}{(3x+1)\sqrt{(3x+1)^2-1}}(x^3-5x^2+1)}{\left(\sec^{-1}(3x+1)\right)^2}$$

Ex: Determine $\frac{dy}{dx}$ for the expression: $\sin^3\left(x^2+5y^3\right)+e^{x^3y^2}=4$

$$\underbrace{3\sin^2(x^2+5y^3)\cos(x^2+5y^3)}_{A}\left(2x+15y^2\cdot y'\right)+\underbrace{e^{x^3y^2}}_{B}\left(3x^2y^2+2y\cdot y'\cdot x^3\right)=0$$

$$y'\left[15y^2A+2x^3yB\right]=-2xA-3x^2y^2B.$$

$$y'=\frac{dy}{dx}=\frac{-2xA-3x^2y^2B}{15y^2A+2x^3yB}$$

2. Anti-derivative: $\Rightarrow$

Anti-derivative of $f(x)$ is $F(x)=?$ such that $F'(x)=f(x)$.

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{derivative}"] & f(x) \arrow[l, bend left, "\text{Anti-derivative}"] \arrow[r, "\text{derivative}"] & f'(x) \end{tikzcd}$$

1. $f(x)=K \Rightarrow F(x)=Kx+C.$
2. $f(x)=x^n \Rightarrow F(x)=\frac{x^{n+1}}{n+1}+C$; if $n\neq -1$
3. $f(x)=x^{-1}=\frac{1}{x} \Rightarrow F(x)=\ln|x|+C.$

$f(x)=\sin x \Rightarrow F(x)=-\cos x+C$

$f(x)=\cos x \Rightarrow F(x)=\sin x+C$

$f(x)=\sec^2 x \Rightarrow F(x)=\tan x+C$

$f(x)=\csc^2 x \Rightarrow F(x)=-\cot x+C.$

$f(x)=e^{kx} \Rightarrow F(x)=\frac{e^{kx}}{k}+C.$

$f(x)=\frac{1}{\sqrt{1-x^2}} \Rightarrow F(x)=\sin^{-1}x+C$

$\vdots$

3. Fundamental Theorem of Calculus (FTC)

FTC part I: $f(x)=\int_a^{u(x)} g(t)\,dt \Rightarrow f'(x)=g(u(x))\cdot u'(x).$

FTC part II. $\int_a^b f(x)\,dx = F(b)-F(a)$

where $F'(x)=f(x).$

Ex: Integrate the following: by u - substitution.. $\int \frac{7x-2}{\sqrt[3]{4x+3}}dx$.

a) $\int \frac{7x-2}{\sqrt[3]{4x+3}}dx$

Let $u = 4x+3$

$du = 4dx$

$\frac{du}{4} = dx$

$x = \frac{u-3}{4}$

$$= \int \frac{7\left(\frac{u-3}{4}\right) - 2}{\sqrt[3]{u}} \cdot \frac{du}{4}$$

$$= \frac{1}{4}\int \frac{\frac{7}{4}u \underbrace{- \frac{21}{4} - 2}_{= -\frac{29}{4}}}{u^{1/3}}\, du = \frac{1}{4}\int \left(\frac{7}{4}u^{2/3} - \frac{29}{4}u^{-1/3}\right) du$$

$$= \frac{1}{16}\left[7\cdot\frac{3}{5}u^{5/3} - 29\cdot\frac{3}{2}u^{2/3}\right] + C.$$

$$= \frac{1}{16}\left[\frac{21}{5}(4x+3)^{5/3} - \frac{87}{2}(4x+3)^{2/3}\right] + C$$

let $u = \sqrt[3]{4x+3}$ → $3u^2du = 4dx$

$u^3 = 4x+3$ → $\frac{3}{4}u^2du = dx$

→ $x = \frac{1}{4}(u^3-3)$

$$= \int \frac{\frac{7}{4}(u^3-3) - 2}{\cancel{u}} \cdot \frac{3}{4}\cancel{u^2}\, u\, du.$$

$$= \frac{3}{4}\int\left(\frac{7}{4}u^3 - \frac{21}{4} - 2\right)u\, du.$$

$$= \frac{3}{4}\int\left(\frac{7}{4}u^4 - \frac{29}{4}u\right)du.$$

$$= \frac{3}{4}\left[\frac{7}{4}\cdot\frac{1}{5}u^5 - \frac{29}{4}\cdot\frac{u^2}{2}\right] + c.$$

$$= \frac{3}{4}\left[\frac{7}{20}\left(\sqrt[3]{4x+3}\right)^5 - \frac{29}{8}\left(\sqrt[3]{4x+3}\right)^2\right] + c$$

b) $\int \left(2+\sqrt{x}\right)^{12}dx$ ✓

let $u = 2+\sqrt{x}$

$u-2 = \sqrt{x}$

$(u-2)^2 = x.$

$2(u-2)du = dx.$

$$\Rightarrow \int u^{12}\cdot 2(u-2)du$$

$$2\int\left(u^{13} - 2u^{12}\right)du = 2\left[\frac{1}{14}\left(2+\sqrt{x}\right)^{14} - \frac{2}{13}\left(2+\sqrt{x}\right)^{13}\right] + C.$$

c) $\int x^8\sqrt[4]{4x^3+1}\,dx = \int x^6\cdot\sqrt[4]{4x^3+1}\cdot x^2dx$

let $u = \sqrt[4]{4x^3+1}$

$u^4 = 4x^3+1$

$4u^3du = 12x^2dx.$

$\frac{1}{3}u^3du = x^2dx$

$x^3 = \frac{1}{4}(u^4-1)$

$x^6 = \frac{1}{16}(u^4-1)^2$

$$= \int \frac{1}{16}(u^4-1)^2\cdot u\cdot\frac{1}{3}u^3du.$$

$$= \frac{1}{48}\int(u^8 - 2u^4+1)\cdot u^4du.$$

$$= \frac{1}{48}\int(u^{12} - 2u^8 + u^4)du.$$

$$= \frac{1}{48}\left[\frac{1}{13}\left(\sqrt[4]{4x^3+1}\right)^{13} - \frac{2}{9}\left(\sqrt[4]{4x^3+1}\right)^9 + \frac{1}{5}\left(\sqrt[4]{4x^3+1}\right)^5\right] + C.$$

$$\int_{e/5}^{e^2/5} \frac{dx}{x\left[\ln(5x)+4\right]^7}$$

let $u = \ln(5x)+4$ $\begin{cases} x = \frac{e^2}{5} \Rightarrow u = \ln\left(5\cdot\frac{e^2}{5}\right)+4 = 2+4 = 6. \\ x = \frac{e}{5} \Rightarrow u = \ln\left(5\cdot\frac{e}{5}\right)+4 = 5 \end{cases}$

$du = \frac{1}{5x}\cdot 5\cdot dx = \frac{1}{x}dx$

$$= \int_5^6 \frac{du}{u^7} = \int_5^6 u^{-7} du = \frac{u^{-6}}{-6}\Big|_5^6 = -\frac{1}{6}\left[\frac{1}{6^6} - \frac{1}{5^6}\right]$$

---

e) $\int \sqrt{2+\sqrt{3x+1}}\,dx$

Let $u = \sqrt{2+\sqrt{3x+1}}$

$u^2 = 2+\sqrt{3x+1}$

$(u^2-2)^2 = \left(\sqrt{3x+1}\right)^2$

$u^4 - 4u^2 + 4 = 3x+1$

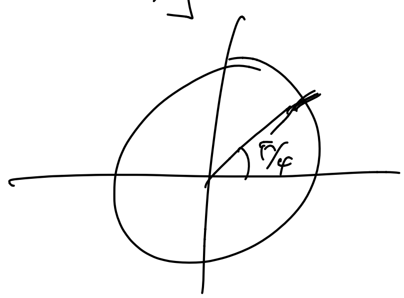
$(4u^3-8u)du = 3dx.$

$\frac{4}{3}(u^3-2u)du = dx.$

$$= \int u\cdot\frac{4}{3}(u^3-2u)du.$$

$$= \frac{4}{3}\int(u^4-2u^2)du.$$

$$= \frac{4}{3}\left[\frac{1}{5}\left(\sqrt{2+\sqrt{3x+1}}\right)^5 - \frac{2}{3}\left(\sqrt{2+\sqrt{3x+1}}\right)^3\right] + C.$$



f) $\int_0^{1/3} \frac{\sqrt[5]{\tan^{-1}(3x)}}{1+9x^2}dx$

Let $u = \sqrt[5]{\tan^{-1}(3x)}$ $\begin{cases} x = \frac{1}{3} \Rightarrow u = \sqrt[5]{\tan^{-1}(1)} = \sqrt[5]{\frac{\pi}{4}} \\ x = 0 \Rightarrow u = 0 \end{cases}$

$u^5 = \tan^{-1}(3x)$

$5u^4 du = \frac{3}{1+9x^2}dx.$

$\frac{5}{3}u^4 du = \frac{1}{1+9x^2}dx$

$$\int_0^{\sqrt[5]{\frac{\pi}{4}}} u\cdot\frac{5}{3}u^4 du. = \frac{5}{3}\int_0^{\sqrt[5]{\frac{\pi}{4}}} u^5 du.$$

$$= \frac{5}{3}\cdot\frac{1}{6}\cdot u^6\Big|_0^{\sqrt[5]{\frac{\pi}{4}}}$$

$$= \frac{5}{18}\left[\left(\sqrt[5]{\frac{\pi}{4}}\right)^6\right] = \cdots = \boxed{\#}$$

Integration by Parts

$$\int (u\cdot v)' = \int u'\cdot v + \int v'\cdot u.$$

$$uv = \int v\,du + \int u\cdot dv.$$

$$\boxed{\int u\cdot dv = uv - \int v\,du.}$$

Take derivative	u	dv	Take anti-derivative
	du	v	

tablet.

$- \int$

Case1: Product of two different types of functions:

- polynomial and exponential functions
- polynomial and sine / cosine
- polynomial and log
- exponential and sine / cosine

short-cut.

Case 2: One function
- log
- *Inverse trig functions*

Case 3: Reduction formulas:

Ex: Integrate the following:

a) $\int \underbrace{(3x-5)}\, e^{2x}dx = \frac{1}{2}e^{2x}(3x-5) - \frac{3}{2}\int e^{2x}dx$

$= \underbrace{A} - \frac{3}{4}e^{2x} + C.$

Deri

$3x-5$	$e^{2x}dx$	anti.
$3\,dx$	$\frac{e^{2x}}{2}$	

$-\int$

b) $\int \left(7x^3-5x^2+3\right)e^{3x+2}dx = e^{3x+2}\left[\frac{1}{3}(7x^3-5x^2+3) - \frac{1}{9}(21x^2-10x) + \frac{1}{27}(42x-10) - \frac{1}{81}(42)\right] + C$

Take derivative to Zero

$7x^3-5x^2+3$	(+)	$e^{3x+2}dx$
$21x^2-10x$	(−)	$\frac{1}{3}e^{3x+2}$
$42x-10$	(+)	$\frac{1}{9}e^{3x+2}$
42	(−)	$\frac{1}{27}e^{3x+2}$
0		$\frac{1}{81}e^{3x+2}$

Take anti-derivative.

=

c) $\int \left(2x^2-5x+4\right)\sin(3x)\,dx = -\frac{1}{3}\left(2x^2-5x+4\right)\cos(3x) + \frac{1}{9}(4x-5)\sin(3x) + \frac{4}{27}\cos(3x)$

$+ C.$

$2x^2-5x+4$	(+)	$\sin(3x)\,dx$
$4x-5$	(−)	$-\frac{1}{3}\cos(3x)$
4	(+)	$-\frac{1}{9}\sin(3x)$
0		$\frac{1}{27}\cos(3x)$

d) $\int (4x^3-5x+2)\ln(2x)\,dx = \underbrace{\left(x^4-\frac{5}{2}x^2+2x\right)\ln(2x)}_{A} - \int \left(x^3-\frac{5}{2}x+2\right)dx.$

$$\begin{array}{c|c} \ln(2x) & (4x^3-5x+2)\,dx \\ \hline \frac{1}{\cancel{2x}}\cdot\cancel{2}dx = \frac{1}{x}dx & x^4-\frac{5}{2}x^2+2x \end{array}$$

$$A - \left(\frac{1}{4}x^4 - \frac{5}{4}x^2 + 2x\right) + C.$$

Cyclist.

e) $\int e^{3x}\cos(4x)\,dx = \underbrace{\frac{1}{4}e^{3x}\sin(4x)}_{A} - \frac{3}{4}\underbrace{\int e^{3x}\sin(4x)\,dx}.$

$$\begin{array}{rc|cl} \text{der.} & e^{3x} & \cos(4x)\,dx & \text{Anti.} \\ \hline & 3e^{3x}dx & \frac{1}{4}\sin(4x) & \end{array}$$

$\sim\int$

$$= A - \frac{3}{4}\left[-\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{4}\int e^{3x}\cos(4x)\,dx\right]$$

$$= A + \frac{3}{16}e^{3x}\cos(4x) - \frac{9}{16}\int e^{3x}\cos(4x)\,dx.$$

$$\begin{array}{c|c} e^{3x} & \sin(4x)\,dx \\ \hline 3e^{3x}dx & -\frac{1}{4}\cos(4x) \end{array}$$

$-\int$

$$\underbrace{\left(1+\frac{9}{16}\right)}\int e^{3x}\cos(4x)\,dx = A + \frac{3}{16}e^{3x}\cos(4x)$$

Ans: $\int e^{3x}\cos(4x)\,dx = \frac{16}{25}\left(A + \frac{3}{16}e^{3x}\cos(4x)\right) + C.$

f) $\int e^{\sqrt[3]{x}}dx = \int e^u\cdot 3u^2\,du. = 3\int u^2 e^u\,du.$

let $u = \sqrt[3]{x}$

$u^3 = x$

$3u^2\,du = dx$

$$\begin{array}{c|c} u^2 & e^u\,du \\ \hline 2u & \boxed{e^u} \\ \hline 2 & \boxed{e^u} \\ \hline 0 & \boxed{e^u} \end{array}$$

(+) (−) (+)

$$= e^u\left(u^2 - 2u + 2\right) + C$$

$$= e^{\sqrt[3]{x}}\left((\sqrt[3]{x})^2 - 2\sqrt[3]{x} + 2\right) + C.$$

g) $\int \cos\left(\sqrt[3]{x}\right)dx = \int \cos(u)\cdot 3u^2\,du = 3\int u^2\cos(u)\,du$

Let $u = \sqrt[3]{x}$

$u^3 = x$

$3u^2\,du = dx$

$$\begin{array}{c|c} u^2 & \cos(u)\,du \\ \hline 2u & \sin(u) \\ \hline 2 & -\cos(u) \\ \hline 0 & -\sin(u) \end{array} \quad (+),\ (-),\ (+)$$

$$= 3\left[\left(\sqrt[3]{x}\right)^2\sin\left(\sqrt[3]{x}\right) + 2\sqrt[3]{x}\cos\left(\sqrt[3]{x}\right) - 2\sin\left(\sqrt[3]{x}\right)\right] + C$$

---

h) $\int \tan^{-1}(5x)\,dx = x\tan^{-1}(5x) - \int \frac{5x}{1+25x^2}\,dx$

Let $u = 1+25x^2$

$\frac{du}{10} = \frac{50x\,dx}{10}$

$\frac{du}{10} = 5x\,dx$

$$\begin{array}{c|c} \tan^{-1}(5x) & dx \\ \hline \frac{5\,dx}{1+25x^2} & x \end{array}$$

Anti-derivative

$-\int$

$$= x\tan^{-1}(5x) - \frac{1}{10}\int \frac{du}{u}$$

$$= x\tan^{-1}(5x) - \frac{1}{10}\ln\left(1+25x^2\right) + C$$

g) $\int \ln(3x+2)\,dx = \underbrace{x\ln(3x+2)}_{A} - \int \frac{3x}{3x+2}\,dx$

Let $u = 3x+2 \Rightarrow 3x = u-2$

$du = 3\,dx$

$\frac{du}{3} = dx$

$$\begin{array}{c|c} \ln(3x+2) & dx \\ \hline \frac{3\,dx}{3x+2} & x \end{array}$$

$-\int$

$$= A - \int \frac{u-2}{u}\cdot\frac{du}{3} = A - \frac{1}{3}\int\left(1 - \frac{2}{u}\right)du$$

$$= A - \frac{1}{3}\left[u - 2\ln|u|\right] + C$$

$$= A - \frac{1}{3}\left[3x+2 - 2\ln|3x+2|\right] + C$$

$$\int \frac{3x+2-2}{3x+2}\,dx = \int\left(1 - \frac{2}{3x+2}\right)dx$$

$$= x - \frac{2}{3}\ln|3x+2| + C$$

$(\tan x)' = \int \sec^2 x\,dx$

Ex. a) Prove the statement: $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$ for $n \neq 1$

b) Use part (a) to evaluate $\int \sec^5(4x)\,dx$

$a^{m+n} = a^m \cdot a^n$

sol for (a): $\int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

$(\sec x)^n = (\sec x)^{n-2} \cdot (\sec x)^2$

$\sec^{n-2}x$	$\sec^2 x\,dx$.
$(n-2)\sec^{n-3}x \cdot \sec x\tan x \cdot dx$	$\tan x$

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$$\Rightarrow \int \sec^n x\,dx = \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n-2}x \cdot \tan^2 x\,dx.$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x\,(\sec^2 x - 1)\,dx.$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx.$$

$$\underbrace{(1+n-2)}_{n-1}\int \sec^n x\,dx = \sec^{n-2}x\tan x + (n-2)\int \sec^{n-2}x\,dx.$$

a) $$\boxed{\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx}$$

b) $\int \sec^5(4x)\,dx$ $\quad \begin{cases} u = 4x \\ \frac{du}{4} = dx. \end{cases}$

$$= \frac{1}{4}\int \sec^5(u)\,du = \frac{1}{4}\left[\frac{\sec^3 u\tan u}{4} + \frac{3}{4}\int \sec^3 u\,du\right]$$

$$= \frac{1}{16}\left[\sec^3 u\tan u + 3\left(\frac{\sec u\tan u}{2} + \frac{1}{2}\int \sec u\,du\right)\right]$$

$$= \frac{1}{16}\left[\sec^3(4x)\tan(4x) + \frac{3}{2}\sec(4x)\tan(4x) + \frac{3}{2}\ln\left|\sec(4x)+\tan(4x)\right|\right] + C$$

Note: $\int \sec^2 x\,dx = \tan x + C$.

??

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.$$

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\cdot dx$$

$$= \int \frac{\sec^2 x + \sec x\tan x\,dx}{\sec x + \tan x}$$

let $u = \sec x + \tan x$

$du = (\sec x\tan x + \sec^2 x)\,dx$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

$$\sec^n x = \sec^{n-2} x \cdot \sec^2 x\,dx$$

$$\int \cos^n x\,dx =$$

$$\int \cos^1 x\,dx = \sin x$$

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx$$

$\cos^{n-1} x$	$\cos x\,dx$
	$\sin x$