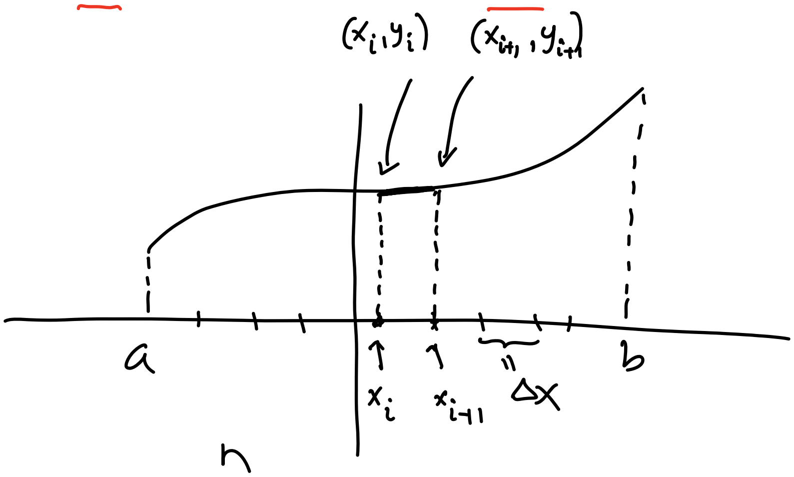
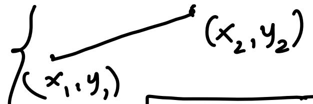
Chapter Eight Further Applications of Integration

Section 8.1 Arc Length

Given a function f(x) is continuous over interval [a,b]. How to find arc – length of f(x) over [a,b]





$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

MVT: $y_2 - y_1 = f'(x_i^*)(x_2 - x_1)$

$$\Rightarrow \text{Length: } \sum_{i=1}^{n} \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (y_{i+1}-y_i)^2}$$

Apply the MVT to $\underset{x_i}{\vdash}\!\!\text{———}\!\!\underset{x_{i+1}}{\dashv}$ $\Rightarrow$ $y_{i+1} - y_i = f'(x_i^*) \cdot \underbrace{(x_{i+1}-x_i)}_{\overset{\shortparallel}{\Delta x}} = f'(x_i^*) \cdot \Delta x$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f'(x_i^*) \cdot \Delta x)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \cdot \Delta x.$$

$$\Rightarrow \text{Arc-length: } \boxed{L = \int_a^b \sqrt{1 + (f'(x))^2} \cdot dx}$$

Def: If f' is continuous on [a, b], then the length of the curve $y = f(x)$ for $a \le x \le b$, is $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Or $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ If the function is in term of x Or $L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ If the function is in term of y.

General form:

$$\text{If } y = f(x) \Rightarrow L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

$$\text{If } x = f(y) \Rightarrow L = \int_c^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy.$$

$L = \int ds$ where $ds = \begin{cases} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx & \text{if } y = f(x) \end{cases}$

$$\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \hat{y} \quad x=f(y)$$

Ex: Find the arc – length of the following:

a) $y = x^{2/3}$ from (1, 1) to (8, 4)

$y = f(x)$

$$L = \int ds = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^8\sqrt{1+\left(\frac{2}{3}x^{-1/3}\right)^2}\,dx.$$

$$= \int_1^8\sqrt{1+\frac{4}{9}x^{-2/3}}\,dx$$

$$= \int_1^8\sqrt{1+\frac{4}{9x^{2/3}}}\,dx.$$

$$= \int_1^8\sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\,dx = \int_1^8\frac{\overset{=u}{\sqrt{9x^{2/3}+4}}}{\underset{=}{3x^{1/3}}}\,\cancel{dx} = \frac{1}{3}\int_{\sqrt{13}}^{\sqrt{40}} \underline{u}\cdot\frac{1}{3}u\,du.$$

let $u = \sqrt{9x^{2/3}+4}$

$u^2 = 9x^{2/3}+4$

$2u\,du = 6x^{-1/3}dx \Rightarrow \frac{1}{3}u\,du = \frac{1}{x^{1/3}}dx,$

$x=8 \Rightarrow u=\sqrt{40}$,

$x=1 \Rightarrow u=\sqrt{13}$

$$= \frac{1}{9}\cdot\frac{1}{3}u^3\Big|_{\sqrt{13}}^{\sqrt{40}} = \frac{1}{27}\left[(\sqrt{40})^3-(\sqrt{13})^3\right]$$

$$= \dots = \#$$

b) $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from x = 1 to x = 2.

$y = f(x)$

$$\Rightarrow L = \int ds = \int_1^2\sqrt{1+(y')^2}\,dx$$

$(a-b)^2 = a^2-2ab+b^2$

$$(y')^2 = \left(\frac{x}{2}-\frac{1}{2x}\right)^2 = \frac{x^2}{4} - \cancel{2}\cdot\frac{x}{\cancel{2}}\cdot\frac{1}{2\cancel{x}} + \frac{1}{4x^2} = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$$

$$1+(y')^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$

$$= \left(\frac{x}{2}+\frac{1}{2x}\right)^2$$

$$L = \int_1^2\cancel{\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}}\,dx = \int_1^2\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \frac{x^2}{4}+\frac{1}{2}\ln|x|\Big|_1^2$$

$$L = 1 + \frac{1}{\ }\ln 2 - \frac{1}{\ } = \boxed{\frac{3}{\ } + \frac{1}{\ }\ln 2} = \#.$$

$2 \qquad 4 \quad \left| \underline{4 \quad 2} \right]$

$y = f(x)$

c) $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right)$ *from* $\underline{x=1}$ *to* $\underline{x=2}$ $\Rightarrow L = \int ds = \int^{2} \sqrt{1+(y')^2}\,dx$

$$y = \ln(e^x+1) - \ln(e^x-1)$$

$$y' = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{e^x(e^x-1) - e^x(e^x+1)}{(e^x+1)(e^x-1)} = \frac{\cancel{e^{2x}} - e^x \cancel{-e^{2x}} - e^x}{e^{2x}-1} = \frac{-2e^x}{e^{2x}-1}$$

$$1+(y')^2 = \left(\frac{-2e^x}{e^{2x}-1}\right)^2 = \frac{4e^{2x}}{e^{4x}-2e^{2x}+1} + 1 = \frac{4e^{2x}+e^{4x}-2e^{2x}+1}{e^{4x}-2e^{2x}+1}$$

$$= \frac{e^{4x}+2e^{2x}+1}{e^{4x}-2e^{2x}+1} = \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2 \Rightarrow L = \int_1^2 \sqrt{\left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2}\,dx$$

$$= \int_1^2 \frac{\overset{u+1}{e^{2x}}+1}{\underbrace{e^{2x}-1}_{=u}}\,dx \qquad \begin{cases} \text{let } \underline{u = e^{2x}-1} \Rightarrow e^{2x} = u+1 \\ du = 2e^{2x}dx = 2(u+1)dx. \\ dx = \dfrac{du}{2(u+1)} \end{cases}$$

$$= \int \frac{u+1+1}{u} \cdot \frac{du}{2(u+1)} = \frac{1}{2}\int \frac{u+2}{u(u+1)}\,du = \frac{1}{2}\int\left(\frac{A}{u} + \frac{B}{u+1}\right)du \qquad \begin{cases} A\big|_{u=0} = \frac{2}{1} = 2 \\ B\big|_{u=-1} = \frac{1}{-1} = -1 \end{cases}$$

$$= \frac{1}{2}\int\left(\frac{2}{u} - \frac{1}{u+1}\right)du = \frac{1}{2}\left[2\ln|u| - \ln|u+1|\right]$$

$$= \frac{1}{2}\left[2\ln|e^{2x}-1| - \ln|e^{2x}-1+1|\right]_1^2 = \dots = \#$$

☆ d) $y = \dfrac{x^3}{3} + x^2 + x + \dfrac{1}{4x+4}$ *for* $0 \le x \le 2$

$y = f(x)$

$$L = \int ds = \int \sqrt{1+(y')^2}\,dx \qquad \begin{cases} y = \frac{1}{3}x^3 + x^2 + x + (4x+4)^{-1} \\ y' = \underbrace{x^2+2x+1}\, - (4x+4)^{-2} \cdot 4. \\ \quad = (x+1)^2 - \dfrac{1}{(4(x+1))^2} \cdot 4 = (x+1)^2 - \dfrac{1}{4(x+1)^2} \end{cases}$$

$$(y')^2 = \left[\underline{(x+1)^2} - \frac{1}{4(x+1)^2}\right]^2 = \overset{a^2 \ -\ 2ab\ +\ b^2}{(x+1)^4 - \underline{\underline{\tfrac{1}{2}}} + \frac{1}{16(x+1)^4}}.$$

$$1+(y')^2 = (x+1)^4 + \frac{1}{2} + \frac{1}{16(x+1)^4} = \left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2.$$

$$\Rightarrow L = \int_0^2 \sqrt{\left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2}\,dx = \int_0^2 \left((x+1)^2 + \frac{1}{4(x+1)^2}\right)dx \qquad \begin{cases} u = x+1 \\ du = dx \end{cases}$$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right)du = \frac{1}{3}u^3 - \frac{1}{4}u^{-1}\Big|_1^3 = \left[9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}\right]$$

$$= \dots = \#.$$

★ e) $x = \frac{y^{3/2}}{3} - y^{1/2}$ from y = $\underline{2}$ to y = 9.

$x = f(y)$

$$L = \int ds = \int_2^9 \sqrt{1 + (x')^2}\, dy$$

$$x = \frac{1}{3} y^{3/2} - y^{1/2} \Rightarrow x' = \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2} \qquad a^2 - 2ab + b^2$$

$$(x')^2 = \left(\frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}\right)^2 = \frac{1}{4} y \underbrace{- \frac{1}{2}} + \frac{1}{4} y^{-1}$$

$$1 + (x')^2 = \frac{1}{4} y + \frac{1}{2} + \frac{1}{4} y^{-1} = \left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2 \leftarrow$$

$$L = \int_2^9 \sqrt{\left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2}\, dy = \int_2^9 \left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right) dy .$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \cdot y^{3/2} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} y^{1/2} \Big|_2^9 = \left(\frac{1}{3} (9)^{3/2} + 9^{1/2}\right) - \left(\frac{1}{3} (2)^{3/2} + 2^{1/2}\right)$$

$$= 9 + 3 - \frac{1}{3}\sqrt{8} - \sqrt{2} = 12 - \frac{2\sqrt{2}}{3} - \sqrt{2}$$

$$= \boxed{12 - \frac{5\sqrt{2}}{3}} = \dots = \#$$

$x = f(y)$

f) $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt \;\; for \; -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

$$L = \int ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + (x')^2}\, dy .$$

$$x = \int_0^y \sqrt{\sec^4 t - 1}\, dt .$$

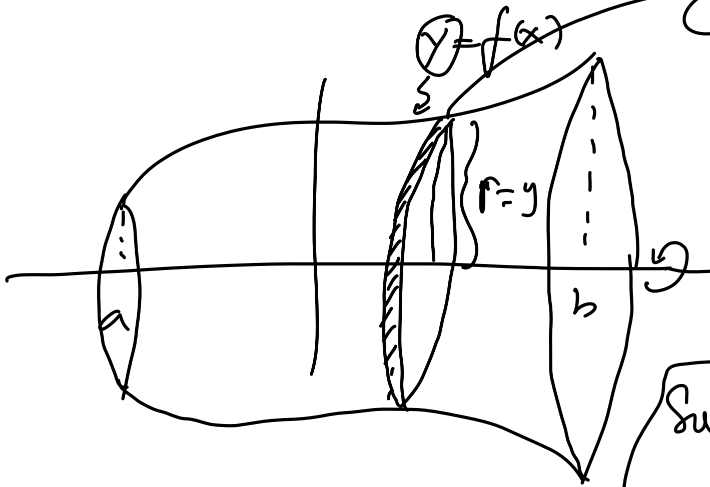
$$x' = \sqrt{\sec^4 y - 1} \Rightarrow (x')^2 = \sec^4 y - 1$$

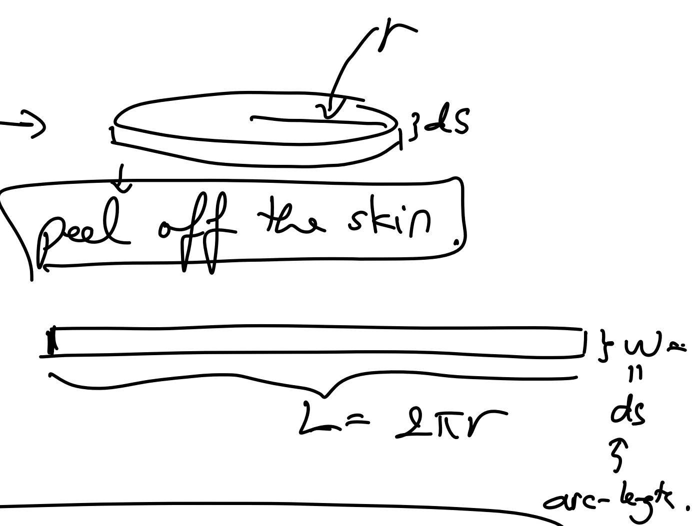
$$1 + (x')^2 = \sec^4 y .$$

$$\Rightarrow L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^4 y}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y\, dy = \tan y \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$L = \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = 1 - (-1) = \boxed{2} .$$

Section 8.2 Area of a Surface of Revolution





Surface-Area: $S = L\cdot W = 2\pi r\,ds$.

$$\boxed{S=\int_a^b 2\pi r\,ds \quad \text{where}}$$

$$ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y=f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x=f(y) \end{cases}$$

$$r = \begin{cases} y & \text{if rotated about the x-axis} \\ x & \text{if ——— y-axis.} \end{cases}$$

Ex: Find the area of the surface formed by rotating about the x axis the arc $y=\dfrac{x^3}{3}$ from x = 0 to x =2.

$$S=\int 2\pi r\,ds = \begin{cases} ds=\sqrt{1+(y')^2}\,dx \quad \text{b/c } y=f(x)=\frac{x^3}{3} \checkmark \\ r=y=\frac{x^3}{3}. \end{cases}$$

$$y' = x^2 \Rightarrow (y')^2 = (x^2)^2 = x^4 \Rightarrow 1+(y')^2 = 1+x^4.$$

$$S = 2\pi\int_0^2 \frac{x^3}{3}\sqrt{1+x^4}\,dx = \frac{2\pi}{3}\int_0^2 \boxed{x^3}\sqrt{1+x^4}\,\boxed{dx}$$

let $u=\sqrt{1+x^4} \Rightarrow \begin{cases} x=2 \Rightarrow u=\sqrt{17}. \\ x=0 \Rightarrow u=1 \end{cases}$

$$\left.\begin{aligned} u^2 &= 1+x^4 \\ 2u\,du &= 4x^3dx \\ \tfrac{1}{2}u\,du &= x^3dx \end{aligned}\right\} \Rightarrow S = \cancel{2}\pi\int_1^{\sqrt{17}} \cdot\, u \cdot \frac{1}{\cancel{2}}u\,du.$$

$$= \pi \cdot \frac{1}{3}u^3\Big|_1^{\sqrt{17}} = \frac{\pi}{3}\left[(\sqrt{17})^3 - 1\right] = \#$$

$$x = f(y)$$

Ex: Find the surface area of the arc of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from (2/3, 1) to (14/3, 3),

a) Rotates about the y – axis.

$$S = 2\pi \int r\,ds \text{ where } \begin{cases} ds = \sqrt{1+(x')^2}\,dy \\ r = x = \frac{1}{6}y^3 + \frac{1}{2y} \end{cases}$$

$$(x')^2 = \left(\frac{1}{2}y^2 - \frac{1}{2}y^{-2}\right)^2 = \frac{1}{4}y^4 - \frac{1}{2} + \frac{1}{4}y^{-4} \Rightarrow 1+(x')^2 = \frac{1}{4}y^4 + \frac{1}{2} + \frac{1}{4}y^{-4} = \left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2$$

$$S = 2\pi \int_1^3 \left(\frac{1}{6}y^3 + \frac{1}{2y}\right)\sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^{\cancel{2}}}\,dy$$

$$= 2\pi \int_1^3 \left(\frac{1}{6}y^3 + \frac{1}{2y}\right)\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)dy = 2\pi \int_1^3 \left(\frac{1}{12}y^5 + \frac{1}{12}y + \frac{1}{4}y + \frac{1}{4}y^{-1}\right)dy$$

$$= 2\pi \left[\frac{1}{72}y^6 + \frac{1}{6}y^2 + \frac{1}{4}\ln|y|\right]_1^3 = 2\pi\left[\frac{3^6}{72} + \frac{9}{6} + \frac{1}{4}\ln 3 - \frac{1}{72} - \frac{1}{6}\right] = \cdots = \#$$

b) Rotates about the x – axis.

$$S = 2\pi \int r\,ds \text{ where } \begin{cases} ds = \sqrt{1+(x')^2}\,dy \Rightarrow 1+(x')^2 = \left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2 \\ r = y \end{cases}$$

$$= 2\pi \int_1^3 y\sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^{2}}\,dy = 2\pi \int_1^3 y\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)dy$$

$$= 2\pi \int_1^3 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-1}\right)dy = 2\pi\left[\frac{1}{8}y^4 + \frac{1}{2}\ln|y|\right]_1^3$$

$$= 2\pi\left[\frac{81}{8} + \frac{1}{2}\ln 3 - \frac{1}{8}\right] = \cdots = \#$$

$y = f(x)$

Ex: The curve $y=\sqrt{2x-x^2}$ for $0.5 \le x \le 1.5$ is rotated about the x – axis. Find its surface area.

$S = 2\pi\int r\,ds$ where $\left\{\begin{array}{l} ds = \sqrt{1+(y')^2}\,dx \quad \text{b/c } y=f(x) \\ r = y = \sqrt{2x-x^2} \quad \text{b/c about the x-axis} \end{array}\right.$ match

✓

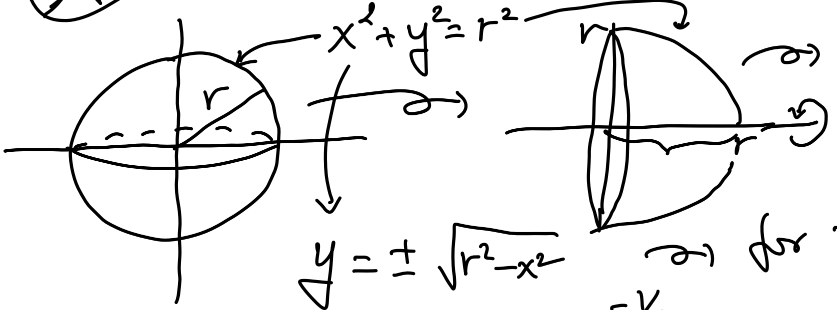
$$y' = \frac{1}{2}\left(2x-x^2\right)^{-\frac{1}{2}}(2-2x) = \frac{\cancel{2}(1-x)}{\cancel{2}\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$$

$$1+(y')^2 = \left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2 = \frac{1-2x+x^2}{2x-x^2}+1 = \frac{1\cancel{-2x}\cancel{+x^2}\cancel{+2x}\cancel{-x^2}}{2x-x^2} = \frac{1}{2x-x^2}$$

$$S = 2\pi\int_{0.5}^{1.5}\underbrace{\sqrt{\cancel{2x-x^2}}\cdot\sqrt{\frac{1}{\cancel{2x-x^2}}}}_{1}\cdot dx = 2\pi\int_{0.5}^{1.5}dx = 2\pi x\Big|_{0.5}^{1.5}$$

$$= 2\pi(1.5-0.5) = \boxed{2\pi}$$

Ex: Show that the surface area of a sphere of radius r is $4\pi r^2$



$\to S = 2\pi\int a\,ds$ $\left\{\begin{array}{l} ds = \sqrt{1+(y')^2}\,dx. \\ a = y = \sqrt{r^2-x^2} \end{array}\right.$ match.

$y = \pm\sqrt{r^2-x^2}$ $\to$ for the Quadrant I $\Rightarrow$ $y = \sqrt{r^2-x^2}$

$$y' = \frac{1}{\cancel{2}}\left(r^2-x^2\right)^{-1/2}(-\cancel{2}x) = \frac{-x}{\sqrt{r^2-x^2}}$$

$$1+(y')^2 = \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2 = \frac{x^2}{r^2-x^2}+1 = \frac{x^2+r^2-x^2}{r^2-x^2} = \frac{r^2}{r^2-x^2}$$

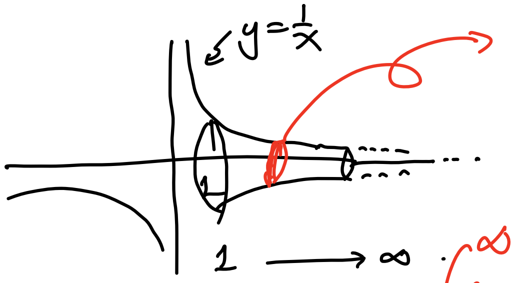
$$S = 2\pi\int a\,ds = 2\cdot 2\pi\int_0^r \sqrt{\cancel{r^2-x^2}}\cdot\sqrt{\frac{r^2}{\cancel{r^2-x^2}}}\cdot dx.$$

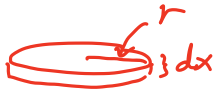
$$= 4\pi\int_0^r r\,dx = 4\pi r x\Big|_0^r = \boxed{4\pi r^2}$$

✓

Ex: The region bounded by $y=\frac{1}{x}$; $y=0$ and $x \geq 1$ is rotated about the x – axis.

a) Find its volume.

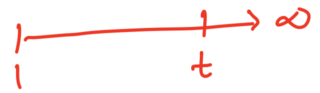




$V = \pi \cdot r^2 \cdot dx$.

where r is $\left.\begin{array}{l} y_{top} = \frac{1}{x} \\ y_{bot} = 0 \end{array}\right\} = r = \frac{1}{x} - 0 = \frac{1}{x}$.

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx .$$

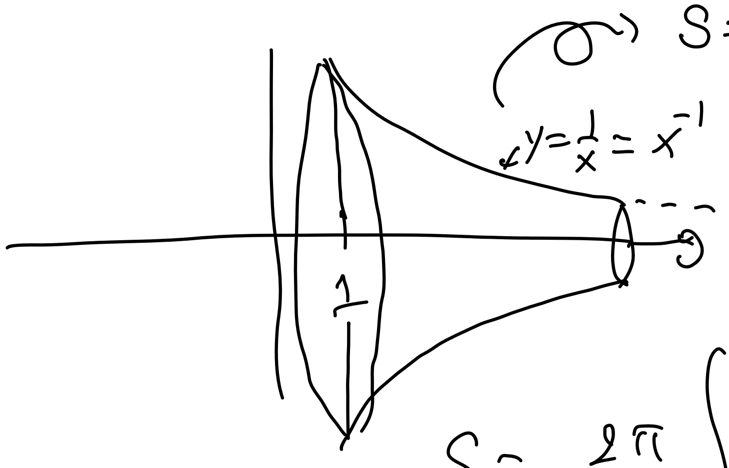


$$= \pi \lim_{t\to\infty} \int_1^t x^{-2}\,dx = \pi \lim_{t\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^t .$$

$$= -\pi \lim_{t\to\infty}\left[\overset{\to 0}{\frac{1}{t}} - 1\right]. = \pi \leftarrow \text{Convergent}$$

b) Find it surface area.

$$S = 2\pi \int r\,ds = 2\pi \int_1^\infty y \sqrt{1+(y')^2}\,dx$$



$$\Rightarrow y' = -x^{-2} = -\frac{1}{x^2} \Rightarrow (y')^2 = \frac{1}{x^4}.$$

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\cancel{\frac{1}{x^4}}}\,dx \geq 2\pi \int_1^\infty \frac{1}{x}\sqrt{1}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x^1}\,dx \quad \left\{ \begin{array}{l} p=1 \\ \text{is divergent} \\ \text{by} \\ \text{p-Test.} \end{array}\right.$$

∴ by C.T.T. S is divergent.

Gabriel's horn problem.

Volume: $V = \pi$ ✓

$S = \underline{\infty}$.