Section 9.3 **Differential Equations** (D.E.)

D.E.

Def: A differential equation is any equation that involves one or more derivative of an unknown function. (i.e. solution(s) of DE are functions)

Where do differential equations come from?

Ex: Solve:

a) $\boxed{y' = y} \Rightarrow \boxed{y = Ke^{x}}$

Population:

Newton's Law of Cooling:

b) $\boxed{y'' = -y} \Rightarrow \begin{cases} y_1 = A\sin x \\ y_2 = B\cos x \\ y = y_1 + y_2 = A\sin x + B\cos x \end{cases}$

Second Newton's Law:

Def: A separable equation is a first – order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. $\rightarrow \frac{dy}{dx} = f(x)\cdot g(y)$

separation of variable

$\int \frac{dy}{g(y)} = \int f(x)\cdot dx$

Ex: Solve the following DE:

a) $\frac{dy}{dx} = \frac{e^{2x-3y}}{e^{2y-3x+2}} = f(x)\cdot g(y)$

$$\frac{dy}{dx} = e^{2x-3y-(2y-3x+2)} = e^{5x-5y-2}.$$

$$\frac{dy}{dx} = \underbrace{e^{5x-2}}_{f(x)} \cdot \underbrace{e^{-5y}}_{g(y)}.$$

$$\frac{b^m}{b^n} = b^{m-n}.$$

$$b^m / b^n = b^{m+n}$$

$$(b^m)^n = b^{m\cdot n}$$

$$\Rightarrow \int \frac{dy}{e^{-5y}} = \int e^{5x-2}\,dx$$

$$\underbrace{\int e^{5y}\,dy} = \underbrace{\int e^{5x-2}\,dx}.$$

$$\frac{1}{5}e^{5y} + C_1 = \frac{1}{5}e^{5x-2} + C_2.$$

$$\frac{1}{5}e^{5y} = \frac{1}{5}e^{5x-2} + (C_2 - C_1) \quad = C$$

$$\frac{1}{5}e^{5y} = \frac{1}{5}e^{5x-2} + C$$

$$e^{5y} = e^{5x-2} + 5C$$

$$e^{5y} = e^{5x-2} + C$$

$$5y = \ln(e^{5x-2} + C)$$

$$\boxed{y = \frac{1}{5}\ln(e^{5x-2} + C)}$$

b) $\dfrac{dy}{dx} = 3x^2y^2 - 5xy^2 + 3y^2 = f(x)\cdot g(y)$

$$\frac{dy}{dx} = y^2(3x^2 - 5x + 3)$$

$$\int \frac{dy}{y^2} = \int (3x^2 - 5x + 3)\,dx$$

$$\int y^{-2}\,dy = \int (3x^2 - 5x + 3)\,dx$$

$$\frac{y^{-1}}{-1} = x^3 - \frac{5}{2}x^2 + 3x + C$$

$$\frac{1}{y} = -x^3 + \frac{5}{2}x^2 - 3x + C$$

$$\boxed{y = \frac{1}{-x^3 + \frac{5}{2}x^2 - 3x + C}}$$

$$\boxed{\frac{dy}{dx} = f(x)\cdot g(y) \qquad \int \frac{dy}{g(y)} = \int f(x)\cdot dx}$$

Ex: Initial Value Problem (IVP)

a) $\dfrac{dy}{dx} = \dfrac{2x+1}{2y};\ y(-2) = -1$

$$\int 2y\,dy = \int (2x+1)\,dx$$

$$y^2 = x^2 + x + C$$

use $y(-2) = -1$ $\left\{ \begin{array}{l} x = -2 \\ y = -1 \end{array} \right.$ ✓

$$(-1)^2 = (-2)^2 + (-2) + C$$

$$1 = 4 - 2 + C$$

$$1 = 2 + C$$

$$C = -1$$

$$y^2 = x^2 + x - 1.$$

$$\boxed{y = \pm\sqrt{x^2 + x - 1}}$$

{2 solutions}

b) $\frac{dy}{dx} = 3y + 2;\ y(1) = 3$ $\begin{cases} x = 1 \\ y = 3 \end{cases}$

$$b^{m+n} = b^m \cdot b^n$$

$$\int \frac{dy}{3y+2} = \int dx.$$

$$\frac{1}{3} \ln|3y+2| = x + C.$$

$$\boxed{\ln|3y+2| = 3x + C}$$

$$3y + 2 = e^{3x+C} = e^{3x} \cdot \underbrace{e^{C}}_{=K} = Ke^{3x}$$

$$3y = Ke^{3x} - \frac{2}{3}$$

$$y = Ke^{3x} - \frac{2}{3}$$

$$3 = Ke^{3} - \frac{2}{3}$$

$$3 + \frac{2}{3} = Ke^3$$

$$\frac{11}{3} = K \cdot e^3$$

$$K = \frac{11}{3e^3}$$

sol: $y = \frac{11}{3e^3} e^{3x} - \frac{2}{3}$

$= \frac{11}{3} e^{3x-3} - \frac{2}{3}$

$= \frac{1}{3}\left[11e^{3x-3} - 2\right]$

$e^{\ln(A)} = A$

Mixing Problems:

Ex. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 20L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

20L/min @ 0.03 kg of salt / L.

20kg of salt

5000L of water

20L/min

sol: Let $A(t)$ be the amount of salt in the tank at time t (in mins).

kg comes in/min

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

$$= (20)(0.03) - \left(\frac{A}{5000}\right)(20)$$

$$\frac{dA}{dt} =$$

IVP. $\boxed{\frac{dA}{dt} = \left(0.6 - \frac{1}{250}A\right) \cdot 1;\ A(0) = 20.}$

$$\int \frac{dA}{0.6 - \frac{1}{250}A} = \int dt$$

$$-250 \ln\left|0.6 - \frac{1}{250}A\right| = t + C$$

$$0.6 - \frac{1}{250}A = e^{-\frac{1}{250}t} \cdot \underbrace{e^{C}}_{=K}$$

$$0.6 - \frac{1}{250}A = Ke^{-\frac{1}{250}t}.$$

$$-\frac{1}{250}A = Ke^{-\frac{1}{250}t} - 0.6$$

$$\boxed{A(t) = Ke^{-\frac{1}{250}t} + 150}$$

$20 \Rightarrow K = -130$

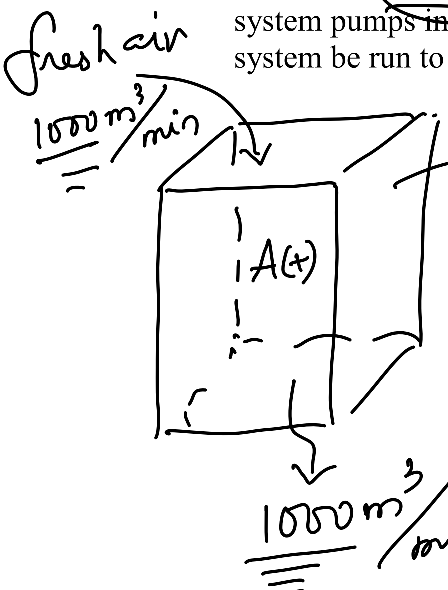
$$\ln\left|0.6 - \frac{1}{250}A\right| = -\frac{1}{250}t + C$$

$A(0) = K + 150 = 20$

$A(t) = -130\, e^{-\frac{1}{250}t} + 150$

After $\frac{1}{2}$ hr $\Rightarrow$ $A(30) = -130\, e^{-\frac{1}{250}(30)} + 150 = 34.7$kg

Ex: A natural gas leak has filled a building enclosing 50,000 m^3 with a 1 percent mixture of natural gas and air. The gas line is shut off, and an emergency ventilation system pumps in fresh air at the rate of 1000 m^3/min. How long must the ventilation system be run to reduce the concentration of natural gas to 0.01 percent?



Volume: $\boxed{50{,}000\ m^3}$

sol: Let $A(t)$ be the amount of natural gas inside the building at time t (in mins).

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}.$$

$$= (1000)(0) - \frac{A}{50,\cancel{000}} \cdot \cancel{1000}.$$

$$\frac{dA}{dt} = 0 - \frac{1}{50}A \;; \quad A(0) = 1\% \text{ of } 50{,}000 = (0.01)(50{,}000) = 500$$

$$\boxed{\frac{dA}{dt} = -\frac{1}{50}A \;; \quad A(0) = 500}$$

$$\int \frac{dA}{A} = -\frac{1}{50}\int dt.$$

$$\ln|A| = -\frac{1}{50}t + C.$$

$$A(t) = e^{-\frac{1}{50}t + C} = e^{-\frac{1}{50}t} \cdot \underbrace{\boxed{e^{C}}}_{=K} = Ke^{-\frac{1}{50}t}.$$

$A(0) = K \cdot e^0 = 500 \Rightarrow K = 500.$

$$\boxed{A(t) = 500\, e^{-\frac{1}{50}t}}$$

$\rightarrow t = ?$ such that $A(t) = 0.01\%$ of $50{,}000 = (0.0001)(50{,}000) = 5$

$A(t) = 5$

$500\, e^{-\frac{1}{50}t} = 5 \Rightarrow e^{-\frac{1}{50}t} = 0.01$

$$e^{-\frac{1}{50}t} = 0.01 \Rightarrow -\frac{1}{50}t = \ln(0.01)$$

$$t = -50 \cdot \ln(0.01) = 230\,\text{min.} \simeq 4\,\text{hr.}$$

Ex: Mortgage: Determine the monthly payment of a loan of \$650,000 at interest rate of 4.5% per year compounded continuously for 30 years. Then determine the total interest of the loan when it's paid off after 30 years. & monthly payment.

Interest / yr

Balance $B(t)$

payment / yr

Sol: let $B(t)$ be the balance of the mortgage at time t (in yrs).

let M be the monthly payment.

$$\frac{dB}{dt} = \underbrace{\text{rate in}} - \underbrace{\text{rate out}}.$$

$$\boxed{\frac{dB}{dt} = 0.045B - 12M,}$$

$$\boxed{B(0) = 650{,}000 \;;\; B(30) = 0}.$$

$$\frac{dB}{dt} = (0.045B - 12M)\cdot 1.$$

$$\underbrace{\int \frac{dB}{0.045B - 12M}} = \underbrace{\int dt}.$$

$$\frac{1}{0.045}\ln|0.045B - 12M| = t + C.$$

$$\ln|0.045B - 12M| = 0.045t + C.$$

$$0.045B - 12M = e^{0.045t}\cdot \underbrace{\boxed{e^{C}}}_{=K} = Ke^{0.045t}$$

$$\frac{0.045B}{0.045} = \frac{Ke^{0.045t}}{0.045} + \frac{12M}{0.045}$$

$\cancel{0.045}$ $\qquad$ $\underbrace{0.045}_{=}$

$$B(t) = Ke^{0.045t} + 266.67M.$$

$$B(0) = K + 266.67M = 650,000.$$

$$K = 650,000 - 266.67M.$$

$$B(t) = (650,000 - 266.67M)e^{0.045t} + 266.67M.$$

$$B(30) = (650,000 - 266.67M)\underbrace{e^{0.045(30)}}_{=3.86} + 266.67M = 0$$

$$2,509,000 - 1029.35M + 266.67M = 0.$$

$$2,509,000 - 762.68M.$$

$$M = \frac{2,509,000}{762.68} = \$3,289.\frac{72}{xx}$$

Total Interest after 30 YRS.

$$(3,289.\tfrac{72}{xx})(12)(30) - 650,000 = \$534,299.\tfrac{20}{xx}$$