<u>***Section 10.1***</u> ***Curves Defined by Parametric Equations***

Def: Let $x = f(t)$ and $y = g(t)$, where f and g are two functions whose common domain is some vertical I. the collection of points defined by $(x, y) = (f(t), g(t))$ is called a plane curve. The equations $x = f(t)$ and $y = g(t)$ where t is in I, are called parametric equations of the curve. The variable t is called a parameter.

Parametric Equation: $\boxed{\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad t \in I}$

t: parameter.

Ex: Sketch the graph of the following parametric equations and indication the direction on the graph.

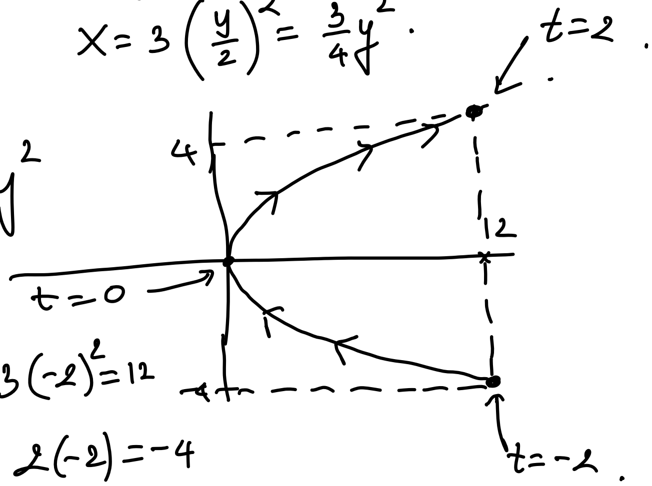
a) $\begin{cases} x = 3t^2 \\ y = 2t \end{cases}$; *for* $-2 \le t \le 2$

Eliminate the parameter t.

$\to t = \frac{y}{2} \to$ plug into $x = 3t^2$

$x = 3\left(\frac{y}{2}\right)^2 = \frac{3}{4}y^2$.

$x = \frac{3}{4}y^2$



$t = -2 \Rightarrow \begin{cases} x = 3(-2)^2 = 12 \\ y = 2(-2) = -4 \end{cases}$

$t = 0 \Rightarrow \begin{cases} x = 3(0)^2 = 0 \\ y = 2(0) = 0 \end{cases}$

$t = 2 \Rightarrow \begin{cases} x = 3(2)^2 = 12 \\ y = 2(2) = 4 \end{cases}$

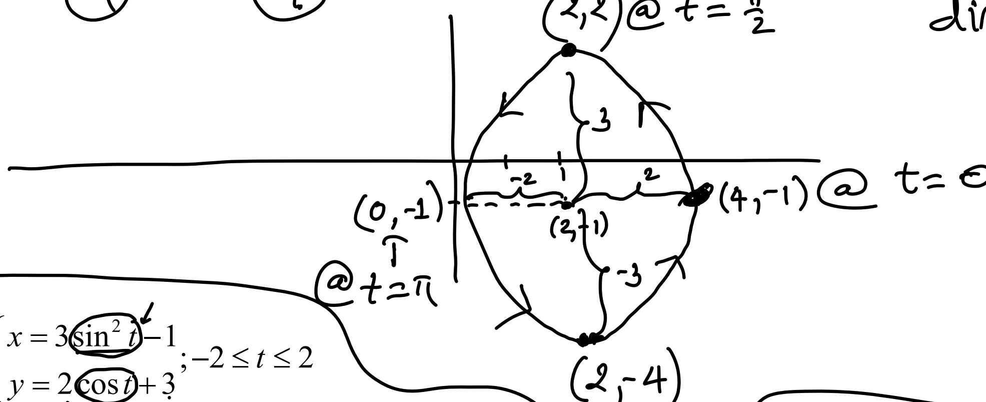
$3t \qquad \frac{}{3} \qquad \frac{t}{3}$

$$\begin{cases} x = 2\cos t + 2 \\ y = 3\sin t - 1 \end{cases} ; \quad t \in \mathbb{R} \text{ (All real numbers).}$$

$$\begin{cases} \cos t = \dfrac{x-2}{2} \\ \sin t = \dfrac{y+1}{3} \end{cases} \xrightarrow[\text{both sides}]{\text{Square}} + \begin{cases} \cos^2 t = \dfrac{(x-2)^2}{4} \\ \sin^2 t = \dfrac{(y+1)^2}{9}. \end{cases}$$

$$1 = \frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} \leftarrow \text{Elipse}.$$

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{9} = 1 \Rightarrow \text{Elipse centered at } (2,-1).$$



direction: $t=0 \Rightarrow \begin{cases} x = 2\cos 0 + 2 \\ = 4 \\ y = 3\sin(0) - 1 \\ = -1 \end{cases}$

$t = \frac{\pi}{2} \begin{cases} x = 2\cos\frac{\pi}{2} + 2 \\ = 2 \\ y = 3\sin\frac{\pi}{2} - 1 \\ = 2 \end{cases}$

$t = \pi \begin{cases} x = 2\cos\pi + 2 \\ = 0 \\ y = 3\sin(\pi) - 1 \\ = -1 \end{cases}$

$$\begin{cases} x = 3\sin^2 t - 1 \\ y = 2\cos t + 3 \end{cases} ; -2 \le t \le 2$$

$$\sin^2 t = \frac{x+1}{3}$$

$$+ \quad \cos^2 t = \left(\frac{y-3}{2}\right)^2$$

$$1 = \frac{x+1}{3} + \frac{(y-3)^2}{4} \quad ?$$

$$3 = x + 1 + \frac{3}{4}(y-3)^2$$

$$x = 3 - 1 - \frac{3}{4}(y-3)^2$$

$$x = -\frac{3}{4}(y-3)^2 + 2 \Rightarrow$$

$x = 3\sin^2 t - 1$

$0 \le \sin^2 t \le 1$

$0 \le 3\sin^2 t \le 3$

$-1 \le 3\sin^2 t - 1 \le 2$

$\boxed{-1 \le x \le 2}$

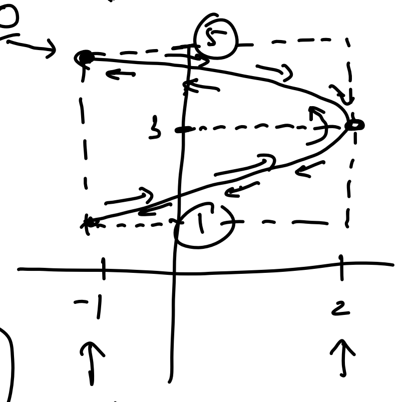
$y = 2\cos t + 3$

$-1 \le \cos t \le 1$

$-2 \le 2\cos t \le 2.$

$1 \le \underbrace{2\cos t + 3} \le 5$

$\boxed{1 \le y \le 5}$



direction:

$t = 0 \Rightarrow \begin{cases} x = -1 \end{cases}$

$y = 5$

d) $\begin{cases} x = (v_0 \cos\theta)t \\ y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h \end{cases}$ (Time as a Parameter: Projectile Motion)

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

gravity $g = \begin{cases} 9.8\ m/sec^2. \\ \text{or} \\ 32\ ft/sec^2. \end{cases}$

$V_0 = 150\ ft/sec$ $\quad V_0 \quad 30°$

Ex: Suppose that Jim hit a golf ball with an initial velocity of 150 ft/sec. at an angle of 30 degree to the horizontal.

a) Find parametric equations that describe the position of the ball as a function of time.

$$\begin{cases} x = f(t) = (V_0 \cos\theta)t = [150 \cos(30°)]t = 150 \cdot \frac{\sqrt{3}}{2} t = 75\sqrt{3}t. \\ y = g(t) = -\frac{1}{2}gt^2 + (V_0 \sin\theta)t + h = -\frac{1}{2}(32) \cdot t^2 + (150 \sin 30)t + 0 \end{cases}$$

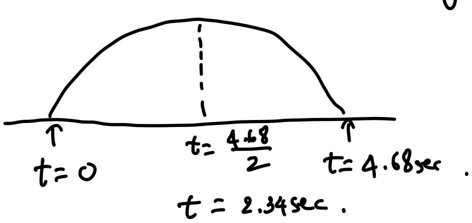
$$= -16t^2 + 75t = y$$

b) How long is the golf ball in the air?

When $y = 0 \Rightarrow -16t^2 + 75t = 0 \Rightarrow t(-16t + 75) = 0 \Rightarrow t = 0,\ \frac{75}{16} = 4.68$ sec.

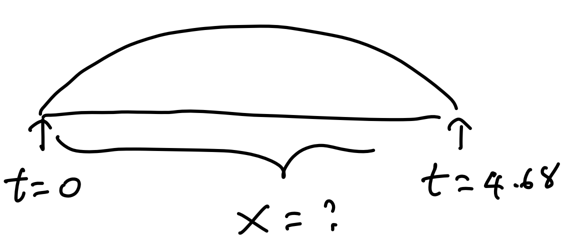
c) When is the ball at its maximum height? Determine the maximum height of the ball.

Max. height: $y\Big|_{t=2.34} = -16t^2 + 75t\Big|_{t=2.34} = -16(2.34)^2 + 75(2.34) = 87.89\ ft$



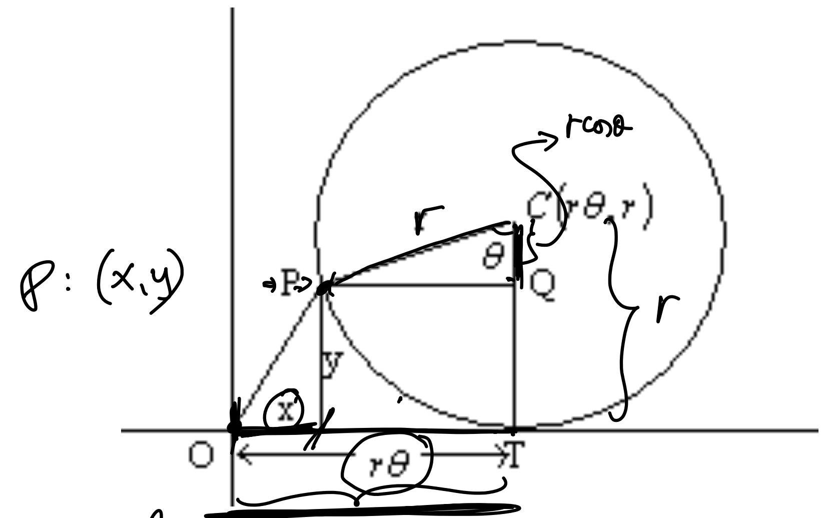
d) Determine the distance that the ball traveled.

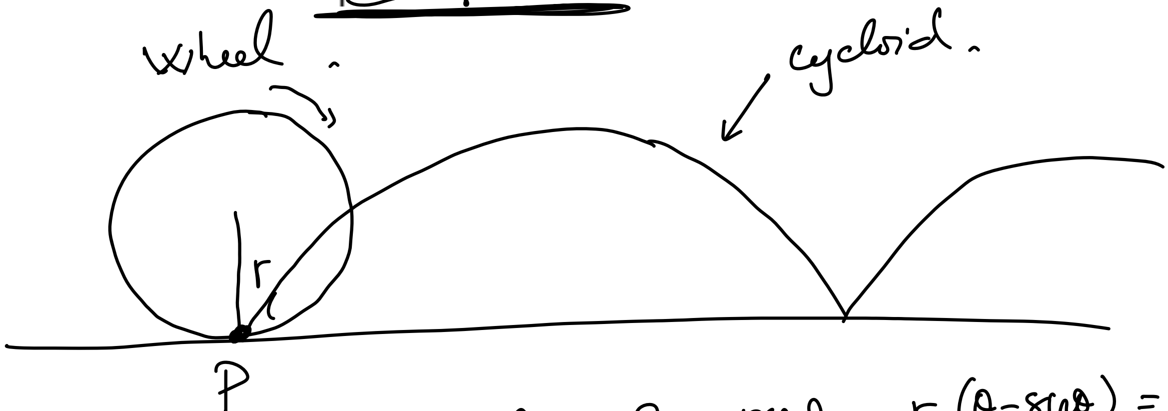
$$x\Big|_{t=4.68} = 75\sqrt{3}t\Big|_{t=4.68} = 75\sqrt{3}(4.68) = 607.95\ ft.$$



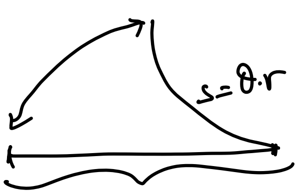
Ex: The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid. If the circle has radius r ~~and rolls~~ along the x-axis and if one position of P is the origin, find the parametric equations for the cycloid.

Sol:





$$\begin{cases} x = \theta r - r\sin\theta = r(\theta - \sin\theta) = f(\theta) \\ y = r - r\cos\theta = r(1-\cos\theta) = g(\theta) \end{cases}$$



cycloid: $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $\theta \in \mathbb{R}$.

9.2

~~Section 11.2~~ *Calculus with Parametric Curves:*

Given a parametric equation: $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ *for* $\boxed{t \in I}$

1. First derivative:

$$y' = ? = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

ex: $\begin{cases} x = 5t^2 + \cos t - 10 \\ y = e^{2t} + 5\sin(3t) + 1 \end{cases} \Longrightarrow y' = \frac{2e^{2t} + 15\cos(3t)}{10t - \sin t} = \frac{g'(t)}{f'(t)}$

$$\frac{d}{dt}(y')$$

2. Second derivative: of $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(y')}{dx/dt} = \frac{\left(\frac{2e^{2t} + 15\cos(3t)}{10t - \sin t}\right)'}{(10t - \sin t)}$$

$$= \frac{(4e^{2t} - 45\sin(3t))(10t - \sin t) - 10(2e^{2t} + 15\cos(3t))}{(10t - \sin t)^3}.$$

Ex: Find point(s) where tangent lines to $\begin{cases} x = t^2 + t - 6 \\ y = t^2 - 3t - 4 \end{cases}$ is either vertical or horizontal

$$\left\{\begin{array}{l} \text{vertical tangent} \Rightarrow y' = \text{undefined} \\ \text{Horizontal} \quad " \quad \Rightarrow y' = 0 \end{array}\right\}$$

$$y' = \frac{dy/dt}{dx/dt} = \frac{2t - 3}{2t + 1} = \frac{a}{b}$$

Vertical tangent $\Rightarrow y' =$ undefined $\Rightarrow 2t + 1 = 0 \Rightarrow t = -\frac{1}{2}$

Point: $\begin{cases} x = t^2 + t - 6 \\ y = t^2 - 3t - 4 \end{cases} \Bigg|_{t = -\frac{1}{2}} = \begin{cases} x = \frac{1}{4} - \frac{1}{2} - 6 = \frac{1 - 2 - 24}{4} \\ x = -\frac{25}{4} \\ y = \frac{1}{4} + \frac{3}{2} - 4 = \frac{1 + 6 - 16}{4} \\ y = -\frac{9}{4} \end{cases}$

$$\left(-\frac{25}{4}, -\frac{9}{4}\right)$$

Horizontal $\Rightarrow y' = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$.

Pt: $\begin{cases} x = t^2 + t - 6 \\ y = t^2 - 3t - 4 \end{cases} \Bigg| = \begin{cases} x = \frac{9}{4} + \frac{3}{2} - 6 = \frac{9 + 6 - 24}{4} = -\frac{11}{4} \\ y = \frac{9}{4} - \frac{9}{2} - 4 = \frac{9 - 18 - 16}{4} = -\frac{25}{4} \end{cases}$ $\left(-\frac{11}{4}, -\frac{25}{4}\right)$

$t=\frac{3}{2}$

Ex: A curve C is defined by the parametric equations $x=t^2; y=t^3-3t$

a) Show that C has two tangents as the point (3,0) and find their equations.

$$\begin{cases} x=f(t)=t^2 \\ y=g(t)=t^3-3t. \end{cases}$$

At point $\begin{cases} x=3 \Rightarrow t^2=3 \Rightarrow t=\pm\sqrt{3} \\ y=0 \Rightarrow t^3-3t=0 \Rightarrow t(t^2-3)=0 \Rightarrow t=0, \pm\sqrt{3}. \end{cases}$

2 tagent line when $t=\pm\sqrt{3}$. $\Rightarrow y' = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2-3}{2t} = m.$

$$m = \left.\frac{3t^2-3}{2t}\right|_{t=\pm\sqrt{3}} = \frac{3(\pm\sqrt{3})^2-3}{2(\pm\sqrt{3})} = \frac{9-3}{\pm 2\sqrt{3}} = \frac{6}{\pm 2\sqrt{3}} = \pm\frac{3}{\sqrt{3}} = \pm\sqrt{3}.$$

There are 2 slopes at (3,0) $\Rightarrow$ C has 2 tangent lines at (3,0).

b) Find the points on C where the tangent is horizontal or vertical.

$y' = m = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2-3}{2t} \Rightarrow \begin{cases} \text{Horizontal} \Rightarrow m=0 \Rightarrow 3t^2-3=0 \Rightarrow t=\pm 1 \\ \text{Vertical} \Rightarrow m=\text{undefined} \Rightarrow 2t=0 \Rightarrow t=0 \end{cases}$

points: Horizontal $\left.\begin{cases} x=t^2 \\ y=t^3-3t \end{cases}\right|_{t=\pm 1} = \begin{cases} x=(\pm1)^2 = 1 \\ y=(\pm1)^3-3(\pm1) = \begin{cases} 1-3=-2 \\ -1+3=2. \end{cases} \end{cases}$ $(1,-2)$ & $(1,2)$

pt Vertical $\left.\begin{cases} x=t^2 \\ y=t^3-3t \end{cases}\right|_{t=0} = \begin{cases} x=0^2=0 \\ y=0^3-3(0)=0 \end{cases}$ $(0,0)$.

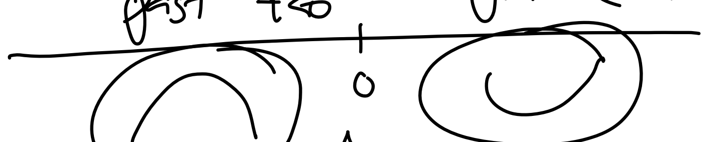
c) Determine where the curve is concave upward or downward.

Concavity $\Rightarrow y'' = \dfrac{\frac{d}{dt}(y')}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \dfrac{\frac{d}{dt}\left(\frac{3}{2}t-\frac{3}{2}t^{-1}\right)}{2t} = \dfrac{\frac{3}{2}+\frac{3}{2t^2}}{2t}$

$y''>0 \Rightarrow \cup$ $\quad$ concave up when $t>0$

$y''<0 \Rightarrow \cap$. $\quad$ concave down when $t<0$.

past $t<0$ $\quad$ future.



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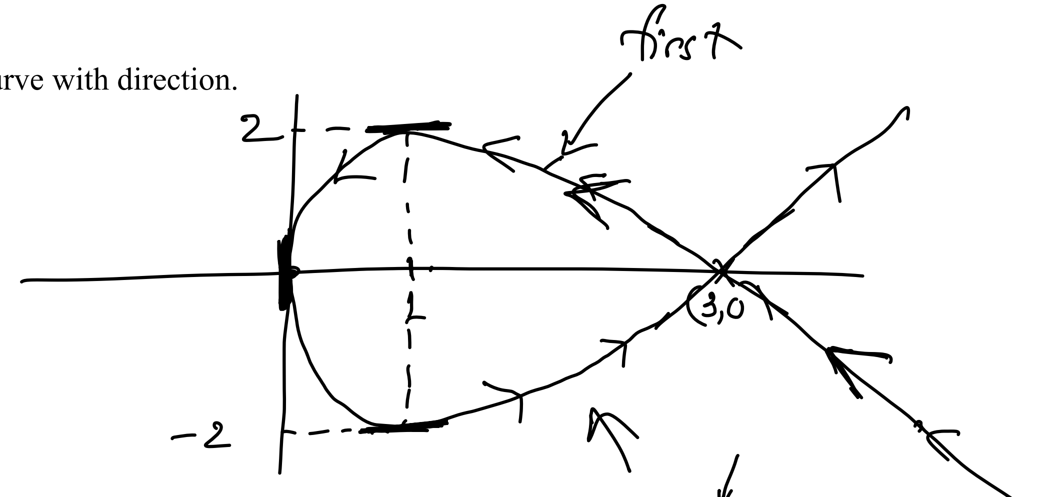
d) Sketch the curve with direction.



Ex: a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$; $y = r(1-\cos\theta)$ at the point $\theta = \pi/3$

↳ in $y = mx + b$

Sol: $y - y_1 = m(x - x_1)$

$$m = y' = \frac{dy/d\theta}{dx/d\theta} = \frac{r(\sin\theta)}{r(1-\cos\theta)}\Bigg|_{\theta=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1-\frac{1}{2}}$$

$$m = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

pt: $$\begin{cases} x_1 = r(\theta - \sin\theta) \\ y_1 = r(1-\cos\theta) \end{cases}\Bigg|_{\theta=\frac{\pi}{3}} = \begin{cases} x_1 = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \\ y_1 = r\left(1 - \cos\frac{\pi}{3}\right) = r\left(1-\frac{1}{2}\right) = \frac{r}{2} \end{cases}$$

$$y - y_1 = m(x - x_1) \Rightarrow \boxed{y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)}$$

b) At what points is the tangent horizontal? When is it vertical?

$$y' = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

$$\text{Area} = \int y\,dx = \int y \cdot \frac{dx}{dt}\,dt$$

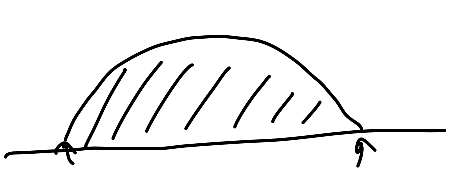
Areas: Given $\begin{cases} x = f(t) \\ y = g(t) \end{cases} \to$ Area for $a \le t \le b$.

$$\text{Area} = \int y\,dx = \int y \frac{dx}{dt}\,dt = \int_a^b g(t)\cdot f'(t)\cdot dt$$

$\begin{cases} x = 2t^2+4 \\ y = t^4-1 \end{cases} \to$ Area: $0 \le t \le 1$

$$\text{Area} = \int y\cdot dx = \int y \boxed{\frac{dx}{dt}} dt = \int_0^1 (t^4-1)(4t)\,dt = 4\int_0^1 (t^5-t)\,dt = 4\left[\frac{1}{6}-\frac{1}{2}\right] = \#$$

Ex: Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$



$\theta = 0$ $\theta = 2\pi$

$$\text{Area} = \int y\,dx = \int_0^{2\pi} y \cdot \frac{dx}{d\theta}\cdot d\theta$$

$$= \int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta = r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$= r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta = r^2\int_0^{2\pi}\left[1 - 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\right]d\theta$$

$$= r^2\int_0^{2\pi}\left[\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\right]d\theta$$

$$= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} = r^2\left[\frac{3}{\cancel{2}}\cdot\cancel{2}\pi\right] = \boxed{3\pi r^2}$$

KNOW This.

$$\boxed{L = \int ds \quad \text{where } ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y = f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x = f(y) \\ \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt & \text{if } \begin{cases} x = f(t) \\ y = g(t) \end{cases} \end{cases}}$$

Arc Length: of $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ for $a \le t \le b$.

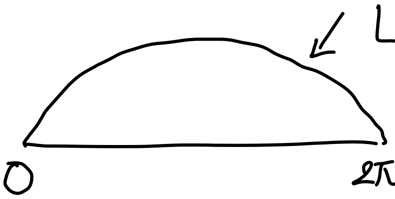
$L = \int ds$ where $ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y = f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x = f(y) \end{cases}$

$$= \int \sqrt{1+(y')^2}\,dx = \int \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\cdot dx = \int \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot\frac{dx}{dt}\cdot dt$$

$$\int \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{\cancel{\frac{dx}{dt}}}\cdot\cancel{\frac{dx}{dt}}\cdot dt$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt}$$

Ex: Find the length of one arch of the cycloid $\begin{cases} x = r(t-\sin t) \\ y = r(1-\cos t) \end{cases}$,



$$L = \int ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$x = r(t-\sin t) \Rightarrow \left(\frac{dx}{dt}\right)^2 = \left(r(1-\cos t)\right)^2 = r^2(1-2\cos t+\cos^2 t)$

$y = r(1-\cos t)$ $\quad \left(\frac{dy}{dt}\right)^2 = \left(r(0+\sin t)\right)^2 = r^2\sin^2 t$.

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 = r^2(1-2\cos t+\underbrace{\cos^2 t+\sin^2 t}_{1})$$

$$= r^2(2-2\cos t) = 2r^2(1-\cos t)$$

$$\Rightarrow L = \int_0^{2\pi}\sqrt{2r^2(1-\cos t)}\,dt = r\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}\,dt.$$

$\sqrt{1-\cos^2 t} = \sqrt{\sin^2 t} = \sin t$.

$$= \sqrt{2}\,r\int_0^{2\pi\ \pi}\frac{\sqrt{1-\cos t}}{1}\cdot\frac{\sqrt{1+\cos t}}{\sqrt{1+\cos t}}\,dt.$$

$$= 2\sqrt{2}\,r\int_0^{\pi}\frac{\sin t}{\sqrt{1+\cos t}}\,dt$$

let $u = \sqrt{1+\cos t}$
$u^2 = 1+\cos t$
$2u\,du = -\sin t\,dt$.
$-2u\,du = \sin t\,dt$.

$x = 0 \Rightarrow u = \sqrt{2}$

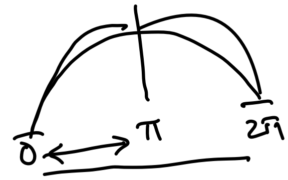
$$= 2\sqrt{2}\,r\int_{\sqrt{2}}^{0}\frac{-2u\,du}{u} = -4\sqrt{2}\cdot r\cdot u\Big|_{\sqrt{2}}^{0}$$

$$= -4\sqrt{2}\,r\,[0-\sqrt{2}]$$

$$= \boxed{8r}$$

$x = \pi \Rightarrow u = 0$

Ex: Find the length of the following curve $\begin{cases} x = \cos t + \ln\left(\tan \dfrac{1}{2}t\right) = f(t) \\ y = \sin t = g(t) \end{cases}$; for $\pi/4 \le t \le 3\pi/4$



$$L = \int ds = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$x = \cos t + \ln\left(\tan\left(\tfrac{1}{2}t\right)\right) \Rightarrow \frac{dx}{dt} = -\sin t + \frac{\sec^2\left(\frac{1}{2}t\right)\left(\frac{1}{2}\right)}{\tan\left(\frac{1}{2}t\right)}$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{2} \cdot \frac{1}{\cos^2\left(\frac{1}{2}t\right)} \cdot \frac{\cos\left(\frac{1}{2}t\right)}{\sin\left(\frac{1}{2}t\right)}$$

$$= -\sin t + \frac{1}{2\cos\left(\frac{1}{2}t\right)\sin\left(\frac{1}{2}t\right)} \overset{?}{=} -\sin t + \frac{1}{\sin t} = \frac{1}{\sin t} - \sin t$$

$$= \sin\left(2 \cdot \tfrac{1}{2}t\right) = \sin t.$$

$$= \frac{1 - \sin^2 t}{\sin t} = \frac{\cos^2 t}{\sin t} \Rightarrow \left(\frac{dx}{dt}\right)^2 = \left(\frac{\cos^2 t}{\sin t}\right)^2 = \frac{\cos^4 t}{\sin^2 t},$$

$$\frac{dy}{dt} = \cos t \Rightarrow \left(\frac{dy}{dt}\right)^2 = \cos^2 t.$$

Surface Area:

$S = 2\pi \int r\, ds$ where

$ds = \begin{cases} \sqrt{1 + (y')^2}\, dx & \text{if } y = f(x) \\ \sqrt{1 + (x')^2}\, dy & \text{if } x = f(y) \\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt & \text{if } \begin{cases} x = f(t) \\ y = g(t) \end{cases} \end{cases}$

$r = \begin{cases} x = f(t) & \text{if rotate about the } y\text{-axis} \\ y = g(t) & x\text{-axis} \end{cases}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{\cos^4 t}{\sin^2 t} + \cos^2 t.$$

$$= \cos^2 t \left[\frac{\cos^2 t}{\sin^2 t} + 1\right].$$

$$= \cos^2 t \left(\cot^2 t + 1\right)$$

$$= \cos^2 t \cdot \csc^2 t.$$

$$\Rightarrow L = \int \sqrt{\cos^2 t \cdot \csc^2 t}\, dt = \int \cos t \cdot \csc t\, dt = \int \frac{\cos t}{\sin t}\, dt = \int \cot t\, dt$$

$$= -\ln|\sin t| \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -\left[\ln\left|\sin\tfrac{3\pi}{4}\right| - \ln\left|\sin\tfrac{\pi}{4}\right|\right],$$

Ex: Find the surface area of the following which rotated about the indicated axis.

a) $\begin{cases} x = e^t - t \\ y = 4e^{t/2} \end{cases}$; $0 \le t \le 1$ rotated about the x – axis.

$$S = 2\pi \int r\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ r = y = 4e^{\frac{t}{2}} \end{cases}$$

match,

$$x = e^t - t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \left(e^t - 1\right)^2 = e^{2t} - 2e^t + 1$$

$$y = 4e^{t/2} \Rightarrow \left(\frac{dy}{dt}\right)^2 = \left(4e^{\frac{t}{2}} \cdot \frac{1}{2}\right)^2 = 4e^t.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1 = \left(e^t + 1\right)^2.$$

$$S = 2\pi \int_0^1 4e^{\frac{t}{2}} \sqrt{\left(e^t + 1\right)^2}\,dt$$

$$= 8\pi \int_0^1 e^{\frac{t}{2}} \left(e^t + 1\right) dt.$$

$$= 8\pi \int_0^1 \left(e^{\frac{3}{2}t} + e^{\frac{1}{2}t}\right) dt,$$

$$= 8\pi \left[\frac{2}{3} e^{\frac{3}{2}t} + 2e^{\frac{1}{2}t}\right]_0^1$$

$$= 8\pi \left[\frac{2}{3} e^{\frac{3}{2}} + 2e^{\frac{1}{2}} - \left(\frac{2}{3} + 2\right)\right] = \cdots = A$$

$$= f(t)$$

b) $\begin{cases} x = \ln(\sec t + \tan t) - \sin t \\ y = \cos t \end{cases}$; $0 \le t \le \frac{\pi}{3}$; *about the* $\boxed{x-\text{axis}}$

$$= g(t)$$

$$S = 2\pi \int r\, ds \quad \text{where} \quad \begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt \\ r = y = \underline{\underline{\cos t}} \end{cases} \quad \text{match.}$$

$$x = \ln(\sec t + \tan t) - \underline{\sin t} \Rightarrow \frac{dx}{dt} = \frac{1}{\sec t + \tan t}\left(\underbrace{\sec t \tan t + \sec^2 t}\right) - \cos t$$

$$\frac{dx}{dt} = \frac{\sec t\,\cancel{(\tan t + \sec t)}}{\cancel{\sec t + \tan t}} - \cos t = \sec t - \cos t = \frac{1}{\cos t} - \cos t$$

$$= \frac{1 - \cos^2 t}{\cos t} = \frac{\sin^2 t}{\cos t} = \tan t \cdot \sin t.$$

$$\left(\frac{dx}{dt}\right)^2 = (\tan t \sin t)^2 = \tan^2 t \sin^2 t.$$

$$+ \quad \left(\frac{dy}{dt}\right)^2 = (-\sin t)^2 = \sin^2 t.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t\,(\underbrace{\tan^2 t + 1}_{= \sec^2 t}) = \underline{\sin^2 t \cdot \sec^2 t}.$$

$$S = 2\pi \int_0^{\pi/3} \cos t \cdot \sqrt{\underline{\sin^2 t} \cdot \underline{\sec^2 t}}\; dt$$

$$= 2\pi \int_0^{\pi/3} \cancel{\cos t} \cdot \sin t \cdot \cancel{\sec t}\, dt.$$

$$= 2\pi \int^{\pi/3} \underline{\sin t}\, dt. = -2\pi \cos t \Big|_0^{\pi/3}$$

$$= -2\pi\left[\underbrace{\cos\frac{\pi}{3}}_{=\frac{1}{2}} - \underbrace{\cos 0}_{=1}\right].$$

$$= -2\pi\left(\frac{1}{2}-1\right) = -2\pi\left(-\frac{1}{2}\right) = \boxed{\pi}$$