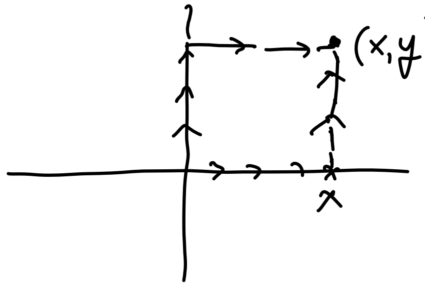
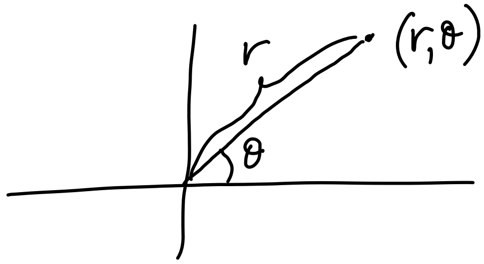
Section 10.2 Polar Coordinates

Def: A point P is represented by the order pair (r, θ) where r is the distance from the point to the origin, and theta is the angle from the x-axis to the line connecting the point and the origin.



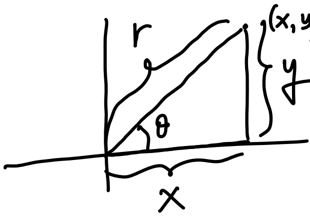


Rectangle Coord.
Cartesian "

Polar - Coord.

So for any point $(x, y) \Rightarrow (r, \theta) \Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

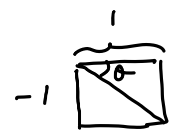
Identities: $x^2 + y^2 = r^2$; $\tan\theta = \dfrac{y}{x}$



$(x, y) = (r, \theta)$

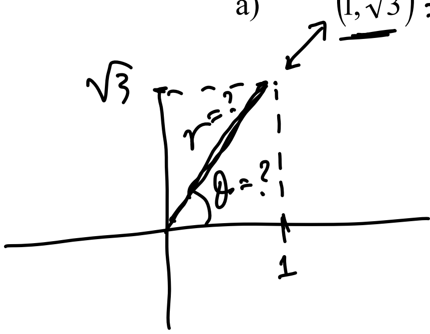
$$\begin{cases} \cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \\ \sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \\ \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

(r, θ)



Ex: Represent the following point in Cartesian coordinate to the polar coordinates.

a) $(1, \sqrt{3}) = (x, y)$

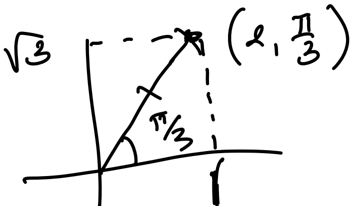


$$x^2 + y^2 = r^2$$
$$(1)^2 + (\sqrt{3})^2 = r^2$$
$$1 + 3 = r^2$$
$$r^2 = 4 \Rightarrow r = \pm 2.$$

Choose $r = 2$, & $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \tan^{-1}(\sqrt{3}) \Rightarrow \theta = \frac{\pi}{3}.$

$(1, \sqrt{3}) \approx \left(2, \frac{\pi}{3}\right).$
$(x, y) \qquad (r, \theta)$



b) $(1, -1)$

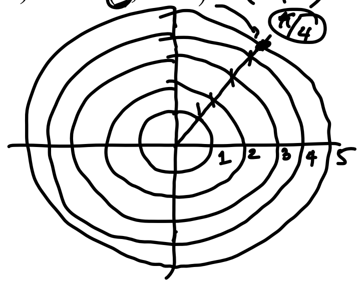
$(1, -1) \approx (r, \theta)$

where $r^2 = x^2 + y^2$
$= 1^2 + (-1)^2 = 2$
$r^2 = 2 \Rightarrow r = \pm\sqrt{2}$
where $\theta = -\frac{\pi}{4} = \frac{7\pi}{4}$

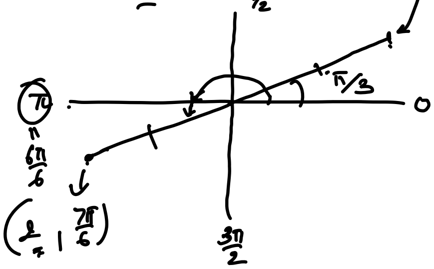
$(1, -1) \approx \left(\sqrt{2}, \frac{7\pi}{4}\right).$

Ex: Locate the following points in polar coordinates.

a) $(5, \pi/4) = (r, \theta)$



b) $(-2, 7\pi/6) = (r, \theta)$ $\left(-2, \frac{7\pi}{6}\right) = \left(2, \frac{\pi}{6}\right)$



$$\begin{cases} x^2+y^2=r^2 \\ x = r\cos\theta \\ y = r\sin\theta \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

xy - coord.

Ex: Convert the following into rectangular coordinates:

a) $r^2 = 3r\sin\theta - 4\cos\theta$

multiply by

$r^3 = 3r^2\sin\theta - 4r\cos\theta$.

$(r^2)^{3/2} = 3r\sin\theta \cdot \sqrt{r^2} - 4r\cos\theta$

$$\boxed{(x^2+y^2)^{3/2} = 3y\sqrt{x^2+y^2} - 4x}$$

b) $r^3 = 2r\cos\theta - 5r\sin\theta$

$(r^2)^{3/2} = 2r\cos\theta - 5r\sin\theta$.

$(x^2+y^2)^{3/2} = 2x - 5y$

c) $\frac{r}{1} = \frac{7}{4\cos\theta - 3\sin\theta}$

$4r\cos\theta - 3r\sin\theta = 7$

$$\boxed{4x - 3y = 7}$$

Ex: Convert to polar coordinate: $\to (r,\theta)$

a) $x^2 + y^2 = 25$

$\Downarrow$

$r^2 = 25$

$\boxed{r = 5}$ ✓

b) $7x - 5y^2 = 4$

$7r\cos\theta - 5(r\sin\theta)^2 = 4.$

$7r\cos\theta - 5r^2\sin^2\theta = 4$

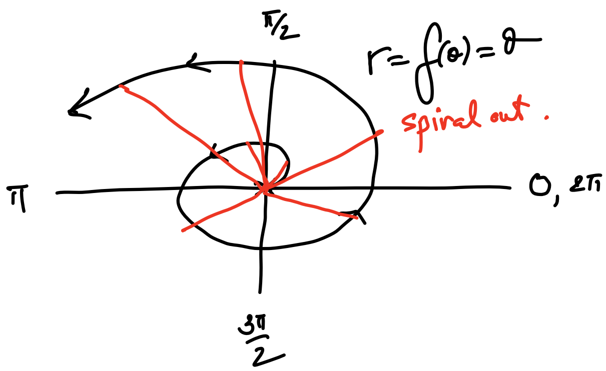
Polar Curves:

The graph of a polar equation $\boxed{r = f(\theta)}$ or more generally, $\boxed{F(r,\theta) = 0}$ consists of all points P that have at least one polar representation (r,θ) whose coordinates satisfy the equation.

Ex: Sketch the graph of the following:

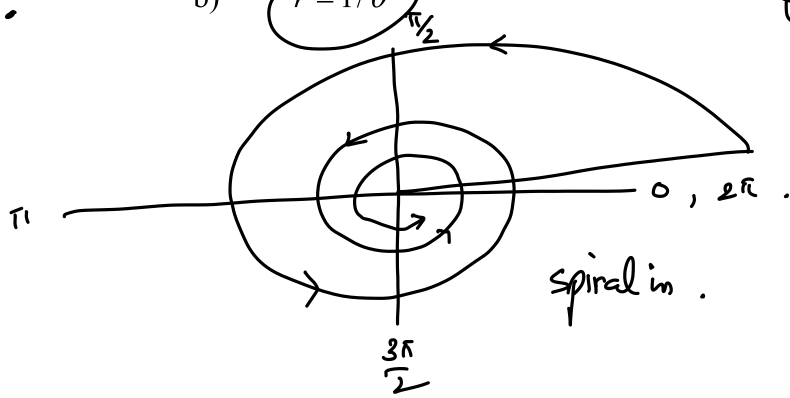
a) $r = \theta$

$r = f(\theta) = \theta$



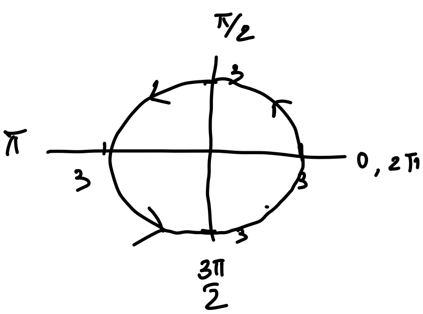
b) $r = 1/\theta$

$\frac{1}{\theta} \leftarrow$ $r = \frac{1}{\theta}$

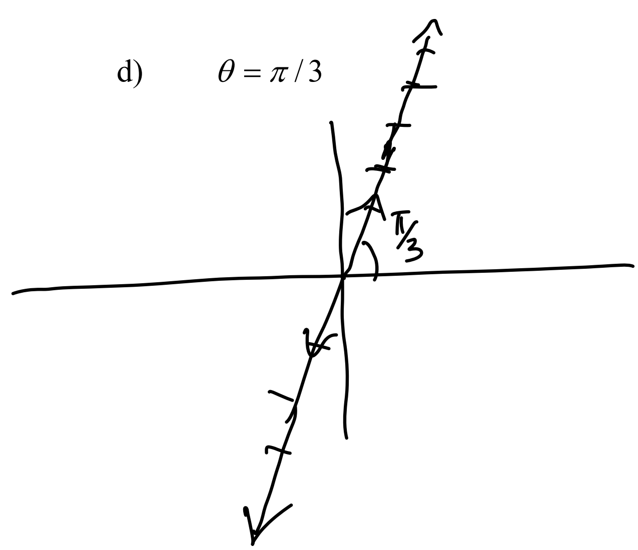


$x = 4$

c) $r = 3$

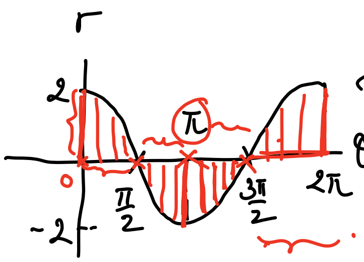


d) $\theta = \pi/3$

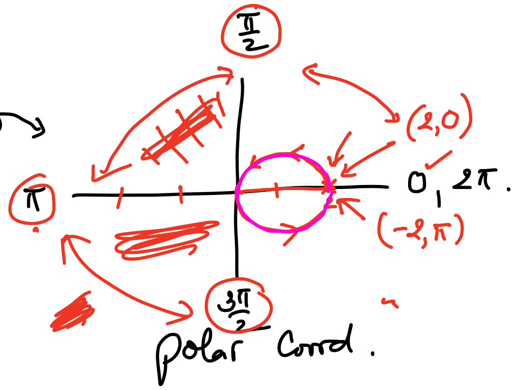


$r = f(\theta) = 2\cos\theta$

e) $r = 2\cos\theta$

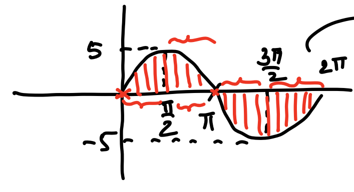


Rectangular

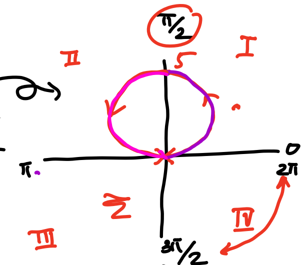


Polar Coord.

f) $r = 5\sin(\theta)$

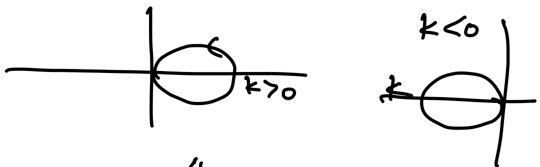


rectangular coord.

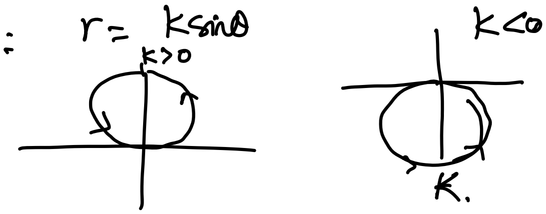


polar coord.

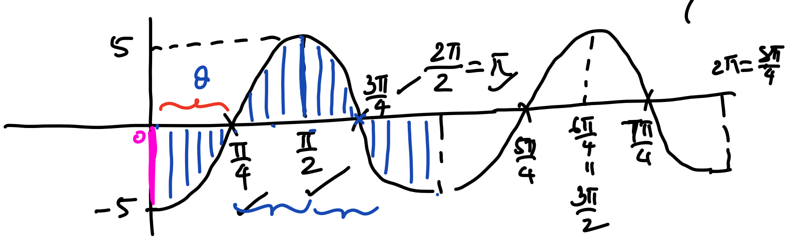
Note: $r = k\cos\theta$, $r =$



Note: $r = k\sin\theta$



l) $r = -5\cos(2\theta)$

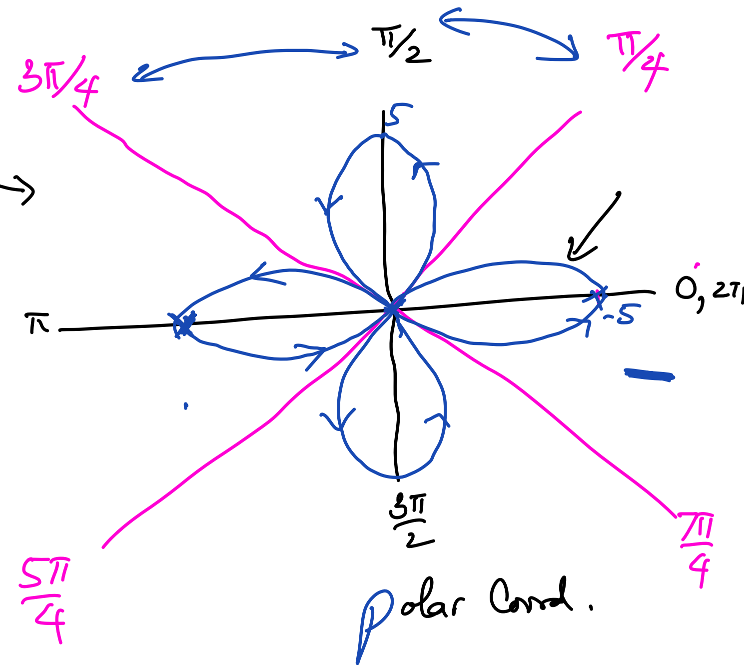


rectangular Coord.

$r = \ldots$

$r^2 = \ldots$

$\pm$



Polar Coord.

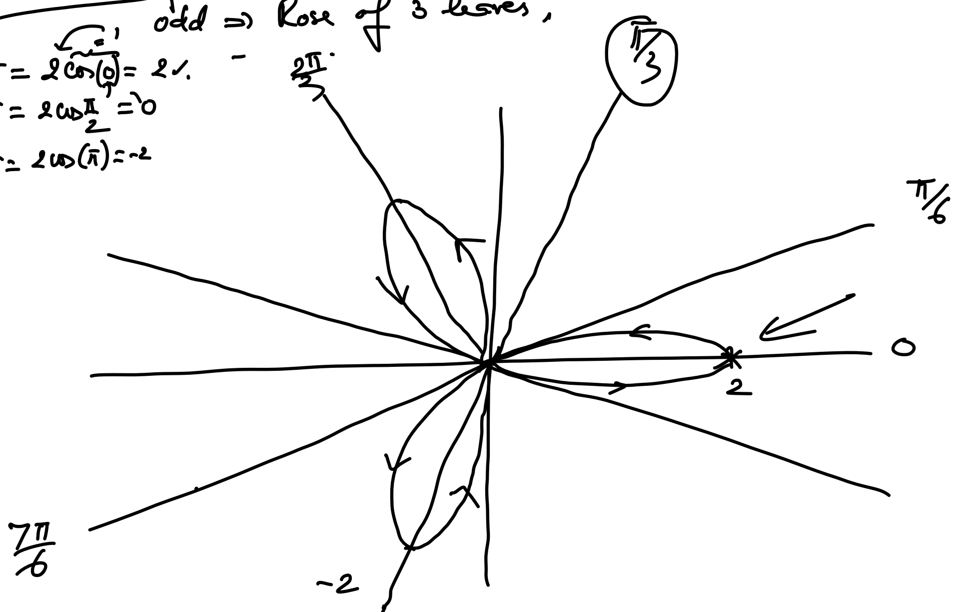
m) $r = 2\cos(3\theta)$

odd ⇒ Rose of 3 leaves,

$\theta = 0 \Rightarrow r = 2\cos(0) = 2$ ✓.

$\theta = \frac{\pi}{6} \Rightarrow r = 2\cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{3} \Rightarrow r = 2\cos(\pi) = -2$



$$r^1 = 7\sin(3\theta)$$

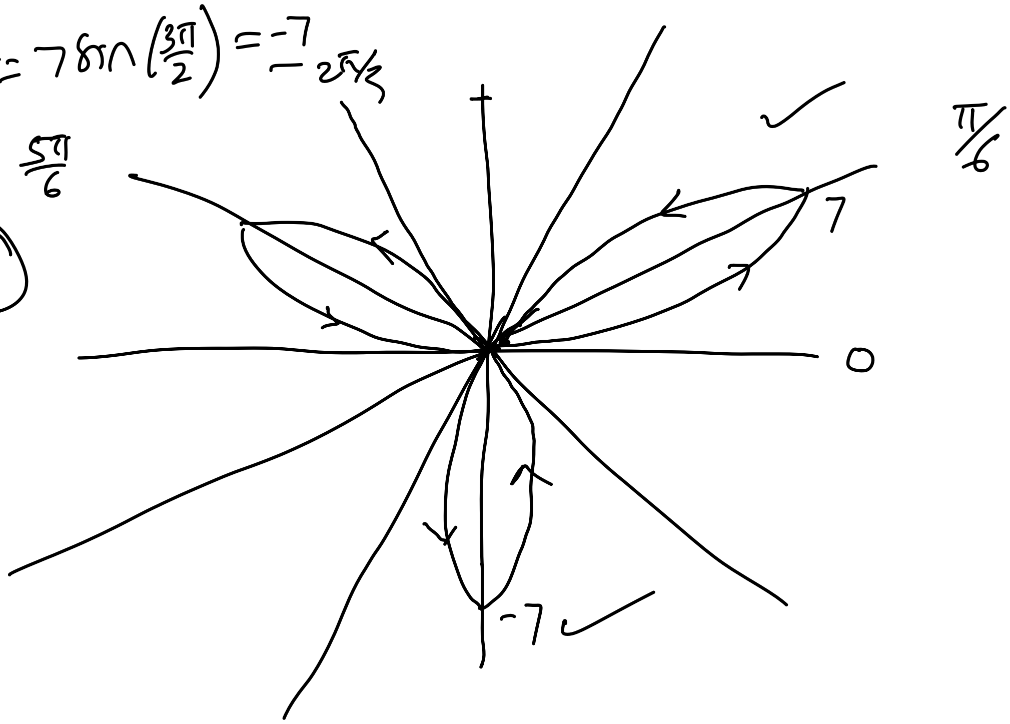
n) $\boxed{r = 3 - 2\cos\theta}$

$\theta = 0 \Rightarrow r = 7\sin(0) = 0$

$\theta = \frac{\pi}{6} \Rightarrow r = 7\sin(3\cdot\frac{\pi}{6}) = 7\sin\frac{\pi}{2} = 7$

$\theta = \frac{\pi}{3} \Rightarrow r = 7\sin\pi = 0$

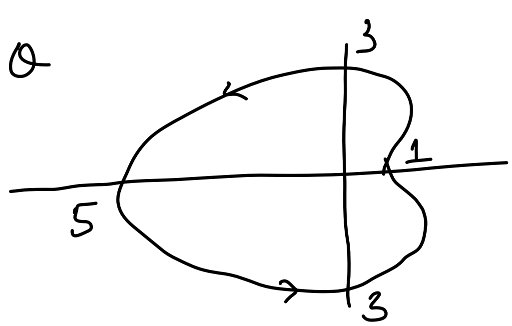
$\theta = \frac{\pi}{2} \Rightarrow r = 7\sin\left(\frac{3\pi}{2}\right) = -7$



o) $r^2 = -9\cos(2\theta)$

a = b
a < b
a > b

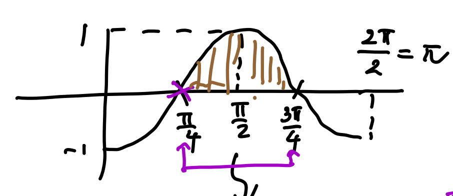
$r = 3 - 2\cos\theta$

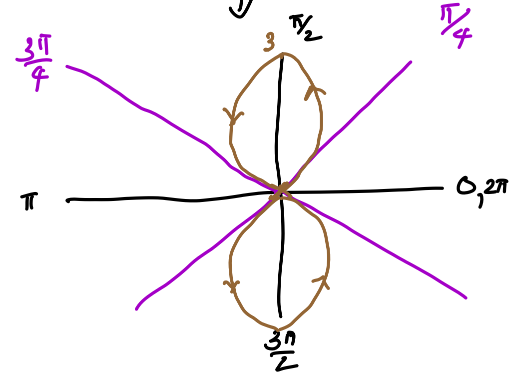


$r^2 = -9\cos(2\theta)$

$r = \pm\sqrt{-9\cos(2\theta)}$

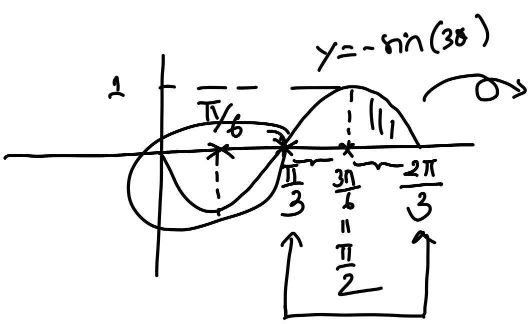
$= \pm 3\sqrt{\boxed{-\cos(2\theta)}} \geq 0$

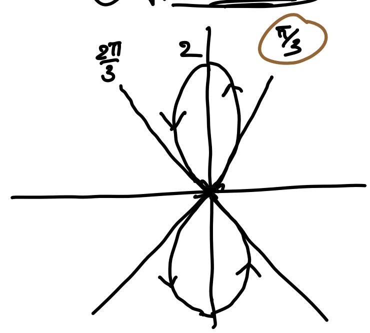




p) $r^2 = -4\sin(3\theta)$

$\Rightarrow r = \pm\sqrt{-4\sin(3\theta)} = \pm 2\sqrt{-\sin(3\theta)}$

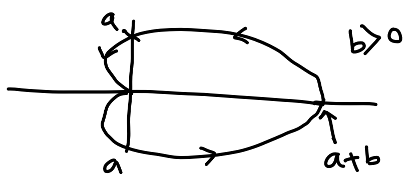


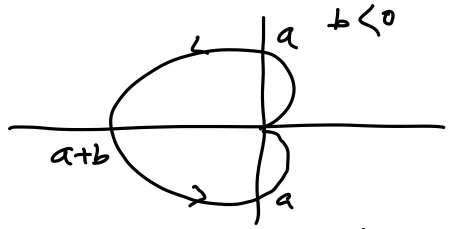


How to sketch $r = a \pm b\cos\theta$ *and* $r = a \pm b\sin\theta$ (Cardiod).

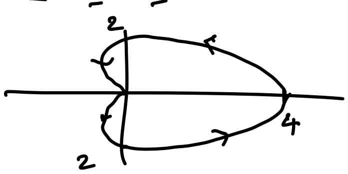
$r = a + b\cos\theta$

Case 1: $a = b$. : perfect heart.

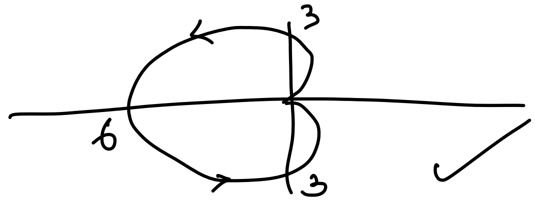




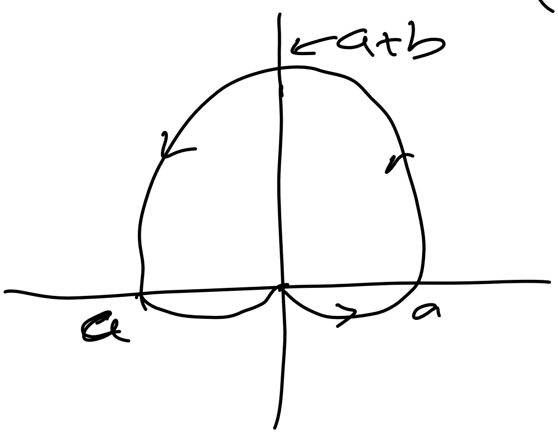
$r = 2 + 2\cos\theta$

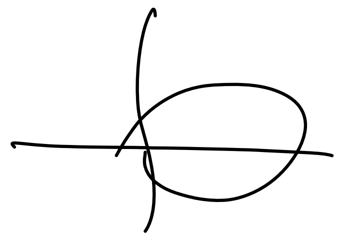


$r = 3 - 3\cos\theta$.

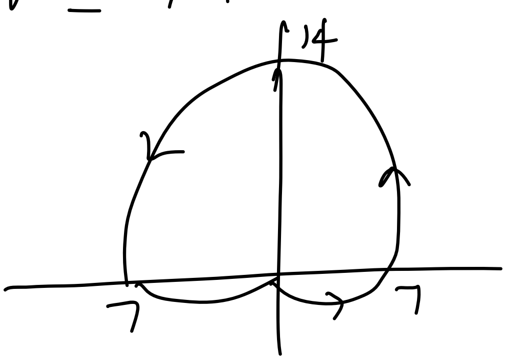


$r = a + b\sin\theta \quad (a = b)$

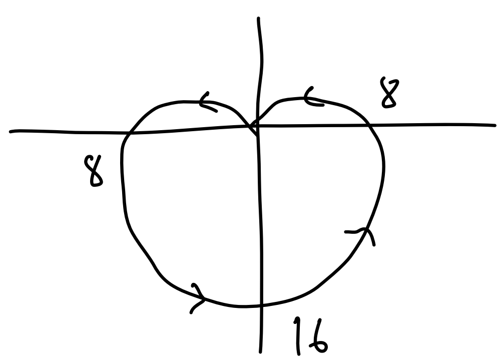




$r = 7 + 7\sin\theta$



$r = 8 - 8\sin\theta$



<u>Tangents to Polar Curves</u>

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

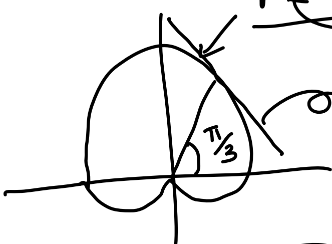
$$x = r\cos\theta = f(\theta)\cos\theta \text{ and } y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\Rightarrow m = y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Ex: a) For the cardioid $\boxed{r = 1+\sin\theta}$, find the slope of the tangent line where $\theta = \pi/3$

$$r = 1+\sin\theta \Rightarrow r' = \cos\theta$$



$\to m = ?$

$$\boxed{m = y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}}$$

$$m = \left.\frac{\cos\theta\cdot\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta}\right|_{\theta=\frac{\pi}{3}}.$$

$$= \frac{\cos\theta\left[\sin\theta + 1 + \sin\theta\right]}{\cos^2\theta - \sin\theta - \sin^2\theta} = \left.\frac{\cos\theta\left[2\sin\theta + 1\right]}{\cos(2\theta) - \sin\theta}\right|_{\theta=\frac{\pi}{3}}$$

$$m = \frac{\cos\frac{\pi}{3}\left[2\sin\frac{\pi}{3} + 1\right]}{\cos\left(\frac{2\pi}{3}\right) - \sin\frac{\pi}{3}} = \frac{\frac{1}{2}\left[\cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} + 1\right]}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} \cdot \frac{(-2)}{(-2)}$$

$$= \frac{-\cancel{(\sqrt{3}+1)}}{\cancel{1+\sqrt{3}}} = \boxed{-1}$$

$\Rightarrow r' = 4\sin\theta$

b) Find the points on the cardioid $r = 4 - 4\cos\theta$ where the tangent line is horizontal or vertical.

Sol:

$$\begin{cases} x = r\cos\theta \Rightarrow \dfrac{dx}{dt} = r'\cos\theta - r\sin\theta \\ y = r\sin\theta \Rightarrow \dfrac{dy}{dt} = r'\sin\theta + r\cos\theta \end{cases}$$

$$m = y' = \frac{dy/dt}{dx/dt} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{4\sin^2\theta + (4-4\cos\theta)\cos\theta}{4\sin\theta\cos\theta - (4-4\cos\theta)\sin\theta}$$

Horizontal $\Rightarrow m = 0 \Rightarrow \cancel{4}\sin^2\theta + \cancel{4}\cos\theta - \cancel{4}\cos^2\theta = 0$.

$$\underbrace{1 - \cos^2\theta + \cos\theta - \cos^2\theta}_{1} = 0$$

$$-2\cos^2\theta + \cos\theta + 1 = 0$$

$$2\cos^2\theta - \cos\theta - 1 = 0.$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0.$$

$$\boxed{\cos\theta = -\frac{1}{2}, 1.}$$

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \begin{cases} \frac{2\pi}{3} + 2n\pi \\ \frac{4\pi}{3} + 2n\pi. \end{cases}$

$\cos\theta = 1$

$\theta = 2n\pi$

Vertical : $m = \text{undefined} \Rightarrow 4\sin\theta\cos\theta - 4\sin\theta + 4\sin\theta\cos\theta = 0$

$$\Rightarrow \underbrace{\sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta}_{} = 0$$

$$\sin\theta\,[2\cos\theta - 1] = 0.$$

$\sin\theta = 0$

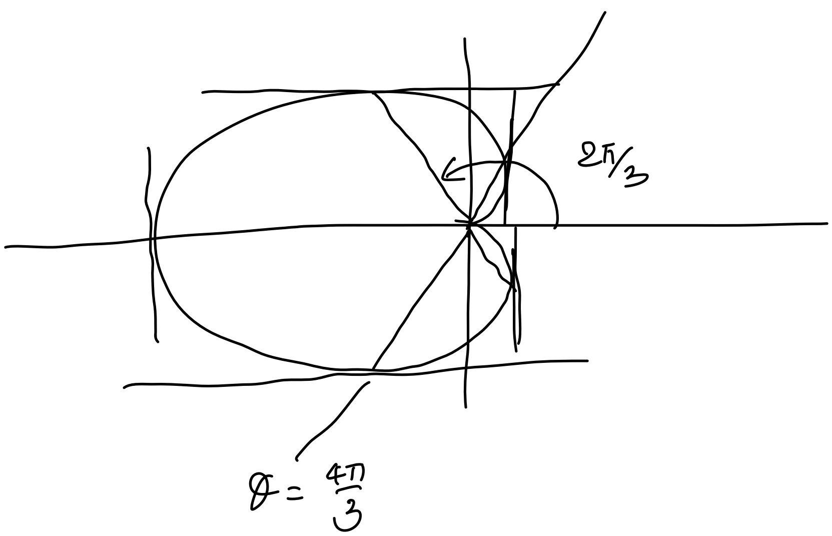
$\cos\theta = \frac{1}{2}$.

$\theta = n\pi$

$$\theta = \begin{cases} \frac{\pi}{3} + 2n\pi \\ \frac{5\pi}{3} + 2n\pi \end{cases}$$

4

$r = 4 - 4\cos\theta$



13). Find slope of tangent line to $r = f(\theta)$ at $\theta = \#$

$m^t = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

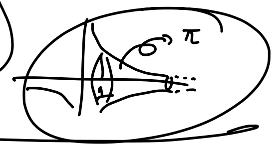
14) Extra - Credit 1

1) surface area of
2) formula of $L = \int ds$
3) Gabriel's horn.

Proof:

1:00pm → 4:20

1:30 → 4:20



$\longrightarrow A =$

$\rightarrow d =$
$\rightarrow v =$
$\rightarrow F =$

?

6) Find L / S.

If $y=f(x)$

$$ds = \begin{cases} 1.\ \sqrt{1+(f')^2}\,dx \\ 2.\ \sqrt{1+(g')^2}\,dy \quad \leftarrow x=f(y) \\ 3.\ \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \end{cases}$$

parametric eq.

Perfect square.

$$\begin{cases} L = \int ds \\ S = 2\pi \int r\,ds \end{cases}$$

$$r = \begin{cases} 1.\ x \text{ about } y\text{-axis} \\ 2.\ y \text{ about } x\text{-axis} \end{cases}$$

$$\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$$ for $0 \le t \le \frac{\pi}{2}$

7) Sketch $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ by

eliminating the parameter t.

- Ellipse.
- Parabola.

8) a) Convert rectangular to polar Coord. $\Leftarrow$

Case 1, 2, 3

b) Sketch $\left(\begin{array}{l} r = a \pm b\cos\theta \\ r = a \pm b\sin\theta \end{array}\right)$ Cardioid,

c) $\boxed{r^2 = -4\cos(2\theta)}$ will be on exam.

$r = \pm\sqrt{-4\cos(2\theta)}$

$\begin{cases} x = e^{t}\cos t \\ y = e^{t}\sin t \end{cases}$ for $0 \le t \le \frac{\pi}{2}$ $\Rightarrow$ find the arc-length.

Sol: $L = \int ds$ where $ds = \begin{cases} 1.\ \sqrt{1+(y')^2}\,dx \\ 2.\ \sqrt{1+(x')^2}\,dy \\ 3.\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \end{cases}$

$x = e^t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \left(e^t(\cos t - \sin t)\right)^2 = e^{2t}\left(\cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t\right)$

$y = e^t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = \left(e^t(\sin t + \cos t)\right)^2 = e^{2t}\left(\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t\right)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}\left[1 + 1\right]$

$= 2e^{2t}.$

$L = \int ds = \int_0^{\frac{\pi}{2}} \sqrt{2e^{2t}}\,dt = \sqrt{2}\int_0^{\pi/2} e^{t}\,dt$

$$= \sqrt{2}\cdot e^{t}\Big|_{0}^{\pi/2}$$

$$= \sqrt{2}\left[e^{\pi/2} - e^{0}\right]$$

$$= \sqrt{2}\left[e^{\pi/2} - 1\right]$$

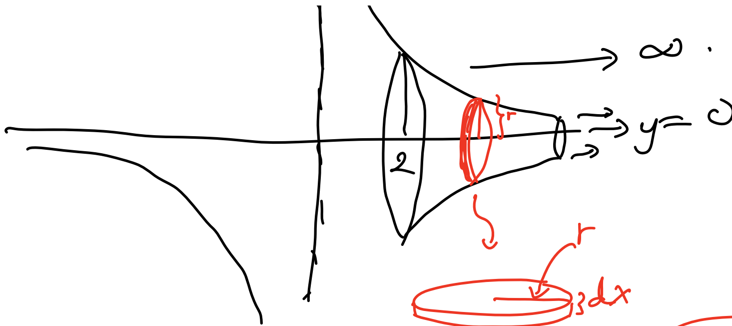
$$\boxed{\begin{cases} x = e^{3t}\cos(2t) \\ y = e^{3t}\sin(2t) \end{cases} \quad 0 \leq t \leq \frac{\pi}{4}.}$$

Ex: The region bounded by $y = \frac{1}{x^5}$ and $y = 0$ for $x \geq 2$. is rotated about the x-axis

$\Rightarrow$ Calculate its volume.

Sol: $y = \frac{1}{x^5}$

$\frac{1}{x}$ $\quad ||$

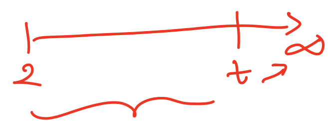


$$V = \pi \cdot r^2 \cdot dx$$

where r is $= y_{top} - y_{bot} = \dfrac{1}{x^5} - 0 = \dfrac{1}{x^5}$

$y_{top} = \dfrac{1}{x^5}$, $\quad y_{bot} = 0$

$$V = \int_2^\infty \pi \left(\frac{1}{x^5}\right)^2 dx$$



$$= \pi \lim_{t\to\infty} \int_2^t \frac{1}{x^{10}}\, dx = \cancel{\ln(x^{10})}$$

$$= \pi \lim_{t\to\infty} \int_2^t x^{-10}\, dx$$

$$= \pi \lim_{t\to\infty} \left[\frac{x^{-9}}{-9}\right]_2^t$$

$$= -\frac{\pi}{9} \lim_{t\to\infty} \left[\underbrace{\frac{1}{t^9}}_{\to 0} - \frac{1}{2^9}\right] = \boxed{\frac{\pi}{9\cdot 2^9}}$$

Convergent

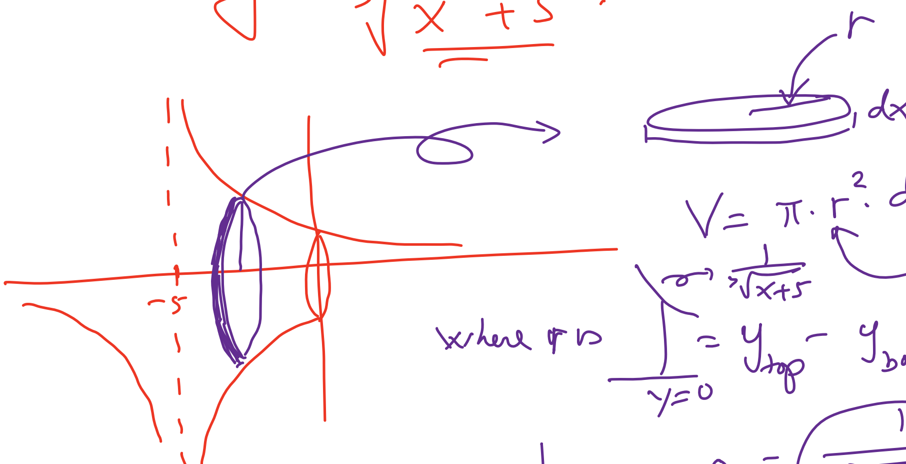
b) The region bounded by $y = \dfrac{1}{\sqrt[3]{x+5}}$, $y=0$

for $-5 \le x \le 0$: about the x-axis

$\Rightarrow$ Find its volume.

Sol:

$$y = \frac{1}{\sqrt[3]{x+5}}.$$



$V = \pi \cdot r^2 \cdot dx.$

where r is $= y_{top} - y_{bot}$

$y=0$

$r = \dfrac{1}{\sqrt[3]{x+5}} - 0 = \dfrac{1}{\sqrt[3]{x+5}}.$

$$\Rightarrow V = \int_{-5}^{0} \pi \left(\frac{1}{\sqrt[3]{x+5}} \right)^2 dx$$

$-5 < t \quad 0$

$$= \pi \lim_{t \to -5^+} \int_t^0 \frac{1}{(x+5)^{2/3}} dx.$$

let $u = x+5$

$du = dx$

$$= \pi \lim_{t \to -5^+} \underbrace{\int \frac{1}{u^{2/3}} du}.$$

7:20

+30

7:50 pm.

$$= \pi \lim_{t\to -5^+} \int u^{-2/3}\,du \,.$$

$$= \pi \lim_{t\to -5^+} u^{1/3}\cdot\frac{3}{1}$$

$$= 3\pi \lim_{t\to -5^+} (x+5)^{1/3}\Big|_t^0$$

$0^{1/3} = 0$

$$= 3\pi \lim_{t\to -5^+}\left[5^{1/3} - (t+5)^{1/3}\right].$$

$$= \boxed{3\sqrt[3]{5}\,\pi} \quad \text{Convergent}$$

Ex: Test for convergence / divergence.

a) $\displaystyle\int_2^{\infty} \frac{\sqrt{5x^5+2x^4+1}}{x^8+4x^5+3}\,dx$ ✓

$5 < 10$

$\frac{1}{5} > \frac{1}{10}$

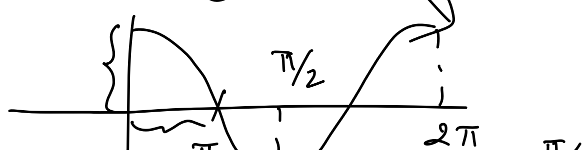
b)

c) $\displaystyle\int_0^{\pi/4} \frac{1}{x^6\cos(2x)}\,dx$

for $0 \le \cos(2x) \le 1$

mult. x^6: $0 \le x^6\cos(2x) \le x^6$

$\frac{1}{x^6\cos(2x)} \ge \frac{1}{x^6}$



$\frac{\pi}{4}$

$$\int_0^{\pi/4} \frac{1}{x^6\cos(2x)}\,dx \geq \int_0^{\pi/4} \frac{1}{x^6}\,dx.$$

0 t $\frac{\pi}{4}$

conv.

$\pm\infty$, DNE

div.

$$\lim_{t\to 0^+}\int_t^{\frac{\pi}{4}} x^{-6}\,dx$$

$$\lim_{t\to 0^+} \frac{x^{-5}}{-5}\Big|_t^{\pi/4}$$

$$= -\frac{1}{5}\lim_{t\to 0^+}\left[\left(\frac{\pi}{4}\right)^{-5} - (t)^{-5}\right]$$

$$\frac{1}{t^5}\to\infty$$

$$=\infty$$

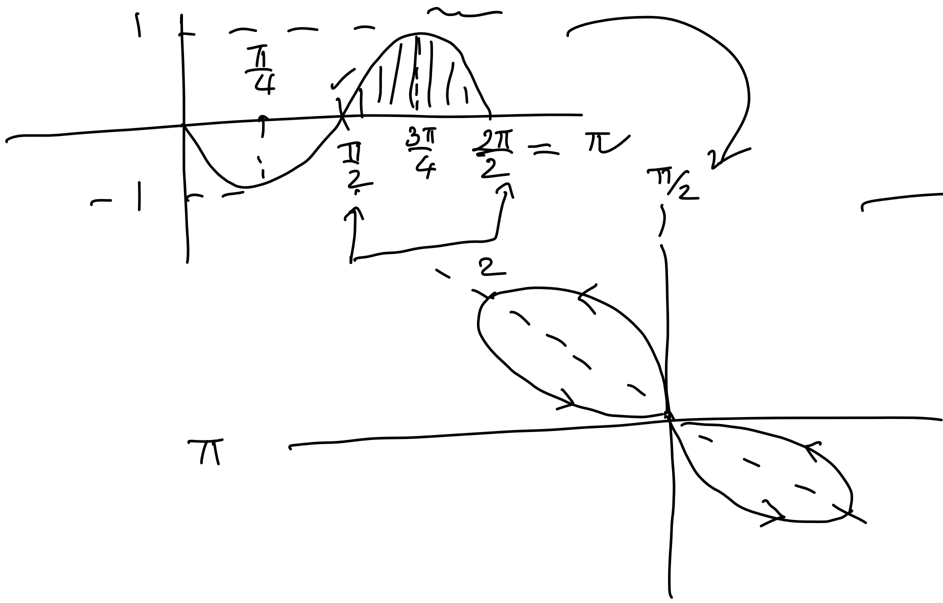
$\therefore$ by C.T.T. $\int_0^{\frac{\pi}{4}} \frac{1}{x^6\cos(2t)}\,dx$ is div.

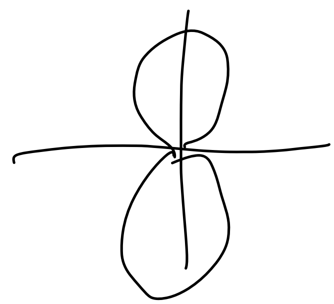
Sketch in polar Coord.

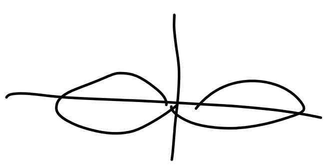
$$r^2 = -4\sin(2\theta)$$

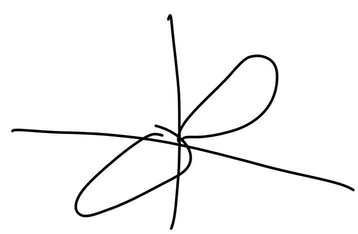
$$r = \pm\sqrt{-4\sin(2\theta)} = \pm 2\sqrt{-\sin(2\theta)}$$

$y = -\sin(2\theta)$







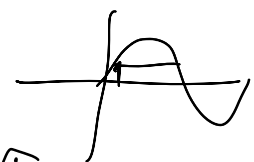


A10) #3b

$$\begin{cases} x = \cos(2t) + t + 2 \\ y = \boxed{2\sin(3t) - 3\sqrt{3}t + 1} \end{cases} \quad t \in \mathbb{R}$$

pts $\Rightarrow$ where tangent | or —

m = und. m = 0.



$$m = y' = \frac{dy/dt}{dx/dt} = \frac{6\cos(3t) - 3\sqrt{3}}{-2\sin(2t) + 1}$$

$2t^2 - 1 = 0 \quad t = \pm\sqrt{\frac{1}{2}}$

Vertical $\Rightarrow$ $-2\sin(2t) + 1 = 0$

$\boxed{\sin(2t) = \frac{1}{2}} \Rightarrow 2t = \begin{cases} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{cases} \Rightarrow t = \begin{cases} \frac{\pi}{12} \\ \frac{5\pi}{12} \end{cases}$

Horizontal $\Rightarrow$ $6\cos(3t) - 3\sqrt{3} = 0$

$$\frac{6\cos(3t)}{6} = \frac{3\sqrt{3}}{6}$$

$\cos(3t) = \frac{\sqrt{3}}{2} \quad 3t = \begin{cases} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{cases} \Rightarrow \begin{cases} \frac{\pi}{18} \\ \frac{5\pi}{18} \end{cases}$