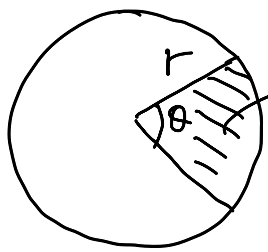
Section 10.4 Area and Lengths in Polar Coordinates : $r = f(\theta)$ in polar coord.



Area: $A = \ ? = \frac{1}{2} r^2 \theta$ ← where θ is in radian.

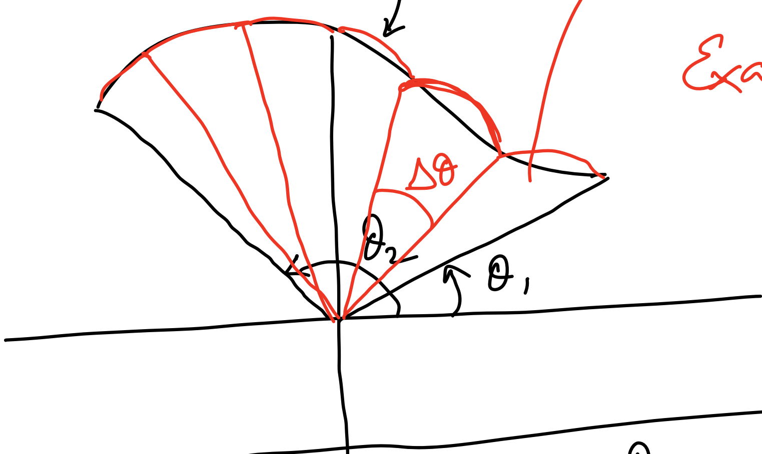
Area Theta

$$\frac{\pi \cdot r^2}{A} = \frac{2\pi}{\theta} \Rightarrow A = \frac{\cancel{\pi} \cdot r^2 \theta}{2\cancel{\pi}} = \frac{1}{2} r^2 \theta$$

Given: $r = f(\theta)$

$$\text{Area} = \sum_{i=1}^{n} \frac{1}{2} \left(f(\theta_i)\right)^2 \cdot \Delta\theta$$

$$\text{Exact area: } \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \left(f(\theta_i)\right)^2 \Delta\theta} = \int_{\theta_1}^{\theta_2} \frac{1}{2} \left(f(\theta)\right)^2 d\theta.$$

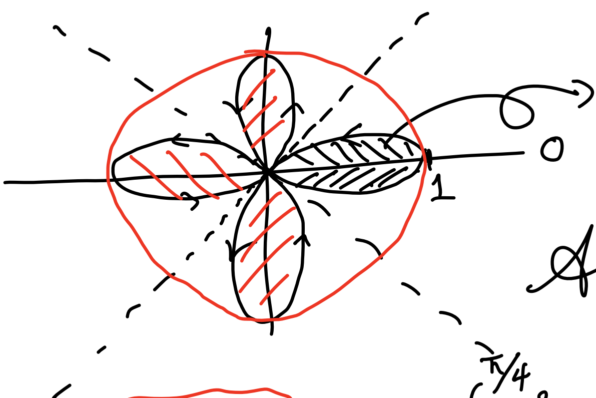


Know this

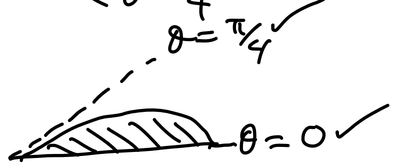
$$\boxed{A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left(f(\theta)\right)^2 d\theta}$$

Ex: Find the area enclosed by one loop of the four-leave rose $r = \cos(2\theta)$.

$r = \cos(2\theta)$



$$\begin{cases} \theta = 0 \Rightarrow r = \cos(0) = 1 \\ \theta = \frac{\pi}{4} \Rightarrow r = \cos\frac{\pi}{2} = 0 \end{cases}$$



$$\text{Area: } \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta = 2\cdot\frac{1}{2}\int_0^{\pi/4} \cos^2(2\theta)\,d\theta.$$

$$A = \int_0^{\pi/4} \cos^2(2\theta)\,d\theta = \int_0^{\pi/4} \frac{1+\cos(4\theta)}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4}$$

$$A = \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8}$$

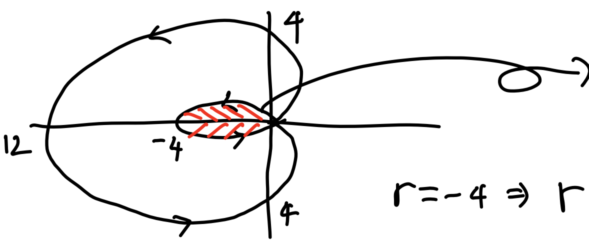
Area of the unit circle $A = \pi\cdot(1)^2 = \pi$.

Area inside all 4 leave

$4\cdot\frac{\pi}{8} = \frac{\pi}{2}$

$$\boxed{A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta}$$

Ex: Find the area of the inner loop of $r = 4 - 8\cos\theta$ =



$r = -4 \Rightarrow r = 4 - 8\cos\theta = -4$

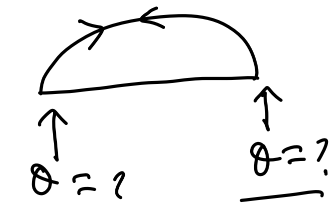
$-8\cos\theta = -8$

$\cos\theta = 1 \Rightarrow \theta_1 = 0$

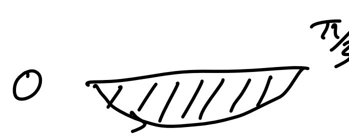
$r = 0 \Rightarrow r = 4 - 8\cos\theta = 0$

$-8\cos\theta = -4$

$\cos\theta = \frac{1}{2} \Rightarrow \theta_2 = \frac{\pi}{3}$



$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta = 2\cdot\frac{1}{2}\int_0^{\pi/3} (4-8\cos\theta)^2\,d\theta$$



$$= \int_0^{\pi/3} \left[16 - 64\cos\theta + 64\cos^2\theta\right] d\theta = \int_0^{\pi/3} 16 - 64\cos\theta + 32(1+\cos(2\theta))\,d\theta.$$

$$= \int_0^{\pi/3} \left[48 - 64\cos\theta + 32\cos(2\theta)\right] d\theta = 48\theta - 64\sin\theta + 16\sin(2\theta)\Big|_0^{\pi/3}$$

$$= 48\left(\frac{\pi}{3}\right) - 64\sin\frac{\pi}{3} + 16\sin\left(\frac{2\pi}{3}\right)$$

$= 16\pi - 64 \cdot \frac{\sqrt{2}}{2} + 16 \cdot \frac{\sqrt{2}}{2} - \pi$

Ex. Given $r_1 = 4\sin(\theta)$ *and* $r_2 = 2$. $\boxed{\text{Set up}}$ integral(s) for area

a) Inside r1 / outside r2

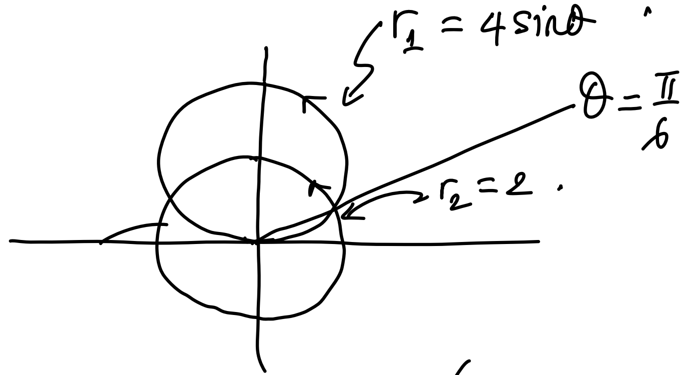
b) Inside r2 / outside r1

c) Inside both r1 and r2.

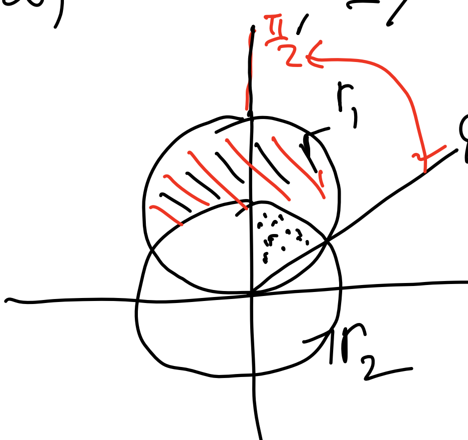
pts of intersection: $\Rightarrow$ solve $\underline{r_1 = r_2}$ for θ.

$\Rightarrow 4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2}$.

$\theta = \frac{\pi}{6}$

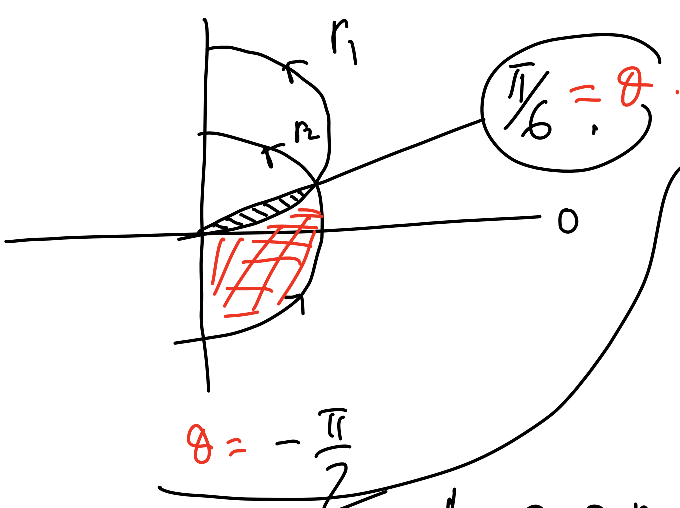


a) Inside r_1 / outside 2

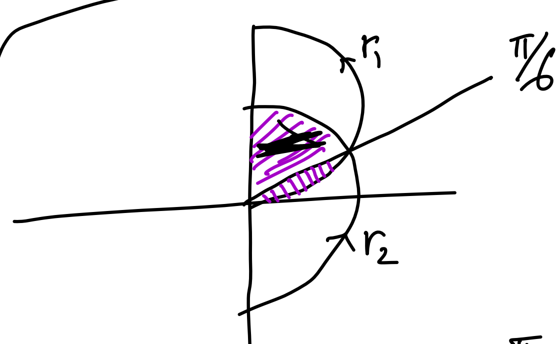


$$\Rightarrow A = \left[\underbrace{\frac{1}{2}\int_{\pi/6}^{\pi/2} (4\sin\theta)^2\, d\theta}_{\text{Inside } r_1} - \underbrace{\frac{1}{2}\int_{\pi/6}^{\pi/2} 2^2\, d\theta}_{\text{Inside } r_2}\right] \cdot 2$$

b) Inside r_2 / outside,

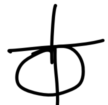


$$A = \left[\underbrace{\frac{1}{2}\int_{-\pi/2}^{\pi/6} 2^2\, d\theta}_{\text{Inside } r_2} - \underbrace{\frac{1}{2}\int_{0}^{\pi/6} (4\sin\theta)^2\, d\theta}_{\text{Inside } r_1}\right] \cdot 2$$



c) Inside both r_1 & r_2

$$A = \left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 2^2\, d\theta + \frac{1}{2}\int_{0}^{\pi/6} (4\sin\theta)^2\, d\theta\right] \cdot 2$$

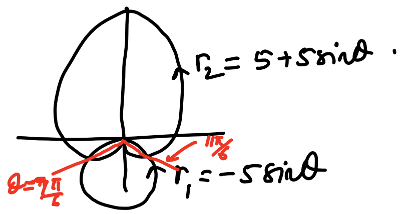
circle  perfect heart 

Ex: Given $r_1 = -5\sin\theta$ *and* $r_2 = 5 + 5\sin\theta$ Sketch and set up integral(s) for area

a) Inside r1 / outside r2.

b) Inside r2 / outside r1

c) Inside both r1 and r2.

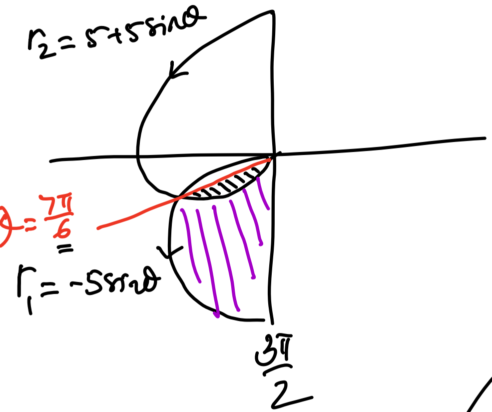


pts of intersections $\Rightarrow r_1 = r_2$

$$-5\sin\theta = 5 + 5\sin\theta \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

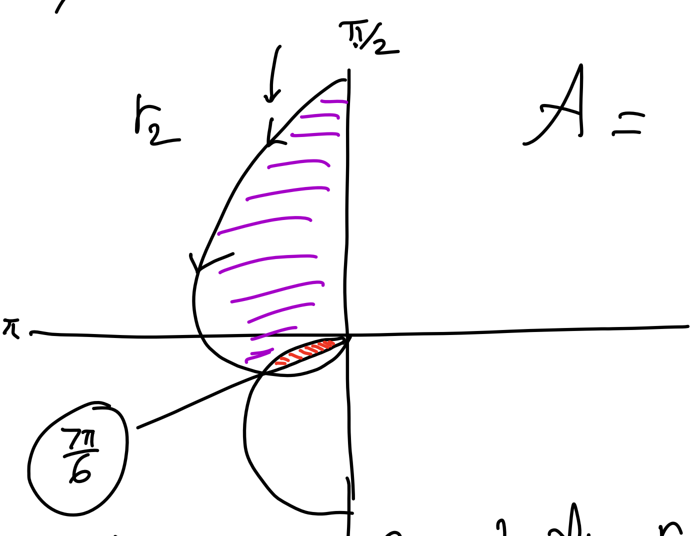
a) Inside r_1 / outside r_2

$$A = \left[\underbrace{\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (-5\sin\theta)^2 d\theta}_{\text{Inside } r_1} - \underbrace{\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (5+5\sin\theta)^2 d\theta}_{\text{Inside } r_2}\right] \cdot 2$$



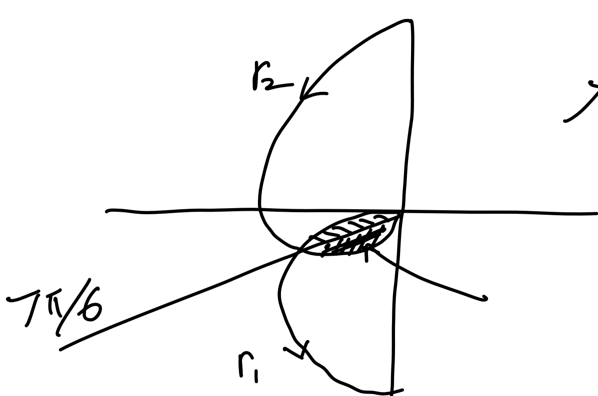
b) Inside r_2 / Outside r_1

$$A = \left[\underbrace{\frac{1}{2}\int_{\pi/2}^{7\pi/6} (5+5\sin\theta)^2 d\theta}_{\text{Inside } r_2} - \underbrace{\frac{1}{2}\int_{\pi}^{7\pi/6} (-5\sin\theta)^2 d\theta}_{\text{Inside } r_1}\right] \cdot 2$$



c) Inside both r_1 & r_2

$$\text{Area: } A = 2\left[\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (5+5\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi}^{7\pi/6} (-5\sin\theta)^2 d\theta\right]$$

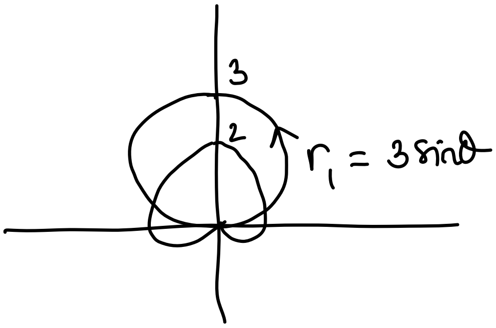


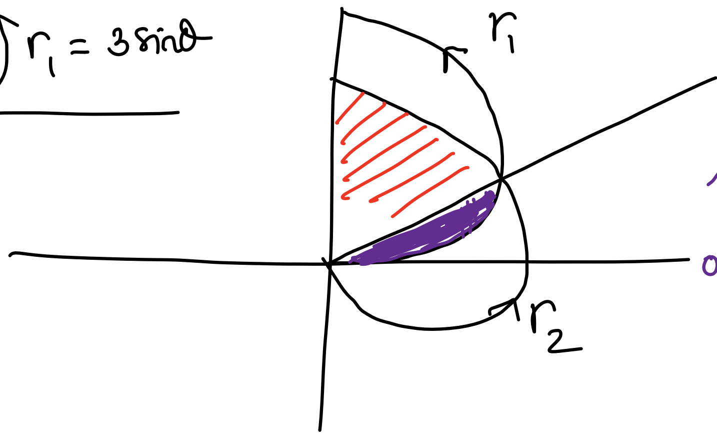
set up

Ex: Find the area of the region that lies inside the circle $r_1 = 3\sin\theta$, and the cardioid $r_2 = 1+\sin\theta$.

pts of intersection $\Rightarrow r_1 = r_2$

$3\sin\theta = 1+\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$



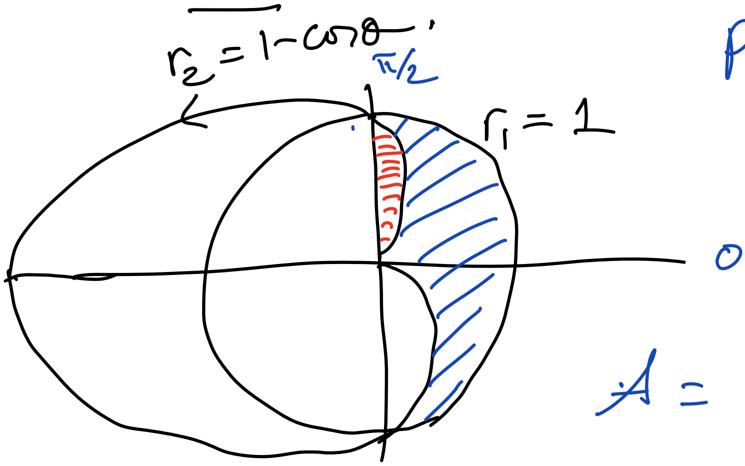


$\pi/6$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/6} (3\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2\, d\theta\right]$$

Ex: Sketch and find the area of the region that lies inside the circle $r = 1$ and outside the cardiod $r = 1-\cos\theta$

1



pts of intersection : $r_1 = r_2$

$1 = 1-\cos\theta \Rightarrow \cos\theta = 0$

$\theta = \frac{\pi}{2}$

$$A = \left[\frac{1}{2}\int_0^{\pi/2} 1^2\, d\theta - \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2\, d\theta\right] 2.$$

Arc – Length for polar coordinate: of $r=f(\theta)$ in polar Coord.

$$\begin{cases} x = r\cos\theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = (r'\cos\theta - r\sin\theta)^2 = (r')^2\cos^2\theta - \cancel{2r'r\cos\theta\sin\theta} + r^2\sin^2\theta \\ y = r\sin\theta \Rightarrow \left(\frac{dy}{d\theta}\right)^2 = (r'\sin\theta + r\cos\theta)^2 = (r')^2\sin^2\theta + \cancel{2r'r\cos\theta\sin\theta} + r^2\cos^2\theta \end{cases}$$

$$L = \int ds = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \boxed{\int_{\theta_1}^{\theta_2} \sqrt{(r')^2 + r^2}\, d\theta}$$

$$L = \int ds = \int \sqrt{(r')^2 + r^2}\, d\theta$$

$$ds = \begin{cases} 1.\ \sqrt{1+(y')^2}\,dx & \text{if } y = f(x) \\ 2.\ \sqrt{1+(x')^2}\,dy & \text{if } x = f(y) \\ 3.\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt & \text{if } \begin{cases} x = f(t) \\ y = g(t) \end{cases} \\ 4.\ \sqrt{(r')^2 + r^2}\,d\theta & \text{if } r = f(\theta) \text{ in polar.} \end{cases}$$

$r = e^\theta$

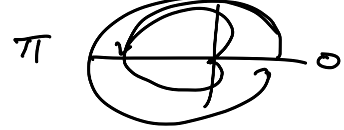
$r' = e^\theta$

Ex: Find the arc – length of the spiral $r = e^\theta$ *for* $0 \le \theta \le \pi$

$$L = \int ds = \int_0^\pi \sqrt{(r')^2 + r^2}\, d\theta = \int_0^\pi \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta.$$

$$= \int_0^\pi \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^\pi \sqrt{2e^{2\theta}}\, d\theta = \sqrt{2}\int_0^\pi e^\theta\, d\theta$$

$$= \sqrt{2}\left[e^\theta\right]_0^\pi = \boxed{\sqrt{2}\cdot\left[e^\pi - 1\right]}$$



Ex: Find the arc – length of the cardioid $r = 1 - \cos\theta$ for $0 \le \theta \le 2\pi$

$$L = \int ds = \int_0^{2\pi} \sqrt{(r')^2 + r^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$(r)^2 = (1-\cos\theta)^2 = 1 - 2\cos\theta + \cos^2\theta$$

$$+ \quad (r')^2 = (\sin\theta)^2 = \sin^2\theta$$

$$(r')^2 + r^2 = 1 - 2\cos\theta + 1 = 2 - 2\cos\theta.$$

$$= \sqrt{2} \int_0^{2\pi} \frac{\sqrt{1-\cos\theta}}{1} \cdot \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\, d\theta = \sqrt{2} \int_0^{2\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta.$$

let $u = \sqrt{1+\cos\theta}$

$$\begin{cases} \theta = \pi \Rightarrow u = 0 \\ \theta = 0 \Rightarrow u = \sqrt{2}. \end{cases}$$

$u^2 = 1 + \cos\theta$

$2u\,du = -\sin\theta\, d\theta$

$-2u\,du = \sin\theta\, d\theta$

$$= 2\sqrt{2} \int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta = -2\sqrt{2} \int_{\sqrt{2}}^{0} \frac{\cancel{u}\,du}{\cancel{u}}.$$

$$= -2\sqrt{2}\, u \Big|_{\sqrt{2}}^{0} = -2\sqrt{2}\left[0 - \sqrt{2}\right] = \boxed{4}$$

Arc-length & Area $\Rightarrow r = f(\theta)$

Area of a surface of revolution: Given $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$, and if the point $P:(r,\theta)$ traces trace the curve $r = f(\theta)$ exactly once for $\alpha \le \theta \le \beta$, then area of the surfaces generated revolving the curve about the x – and y – axes are given

1. Rotated about the x – axis. $S = \int_{\alpha}^{\beta} 2\pi \boxed{r\sin\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad for\ y \ge 0$

   $r = y = r\sin\theta$

2. Rotated about the y – axis $S = \int_{\alpha}^{\beta} 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad for\ x \ge 0$

   $r = \cancel{x}$.

Ex: Find the area of the surface generated by revolving the right – hand loop of the lemniscate $r^2 = \cos(2\theta)$ about the y – axis.