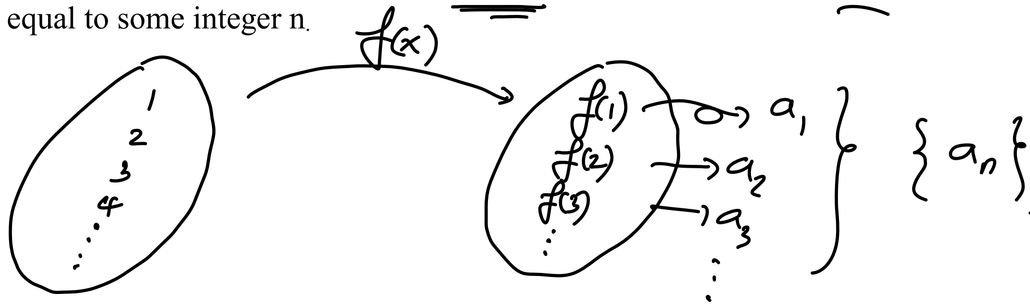
Chapter 11 Infinite Sequences and Series

Section 11.1 ***Sequences***

Def: A sequence can be thought as a list of numbers written in a definite order;
$a_1, a_2, a_3, \ldots, a_n, \ldots$

$\{a_n\} = \{a_1, a_2, a_3, \ldots\} = \{a_0, a_1, a_2, \ldots\}$

Def: An infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer n.



Notation: $\{f(n)\} = \{f(1), f(2), \ldots, f(n), \ldots\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Ex: List the first 4 terms of the following sequences:

a) $\{a_n\} = \left\{\dfrac{2n+1}{n^2+3}\right\} = \left\{\dfrac{3}{4}, \dfrac{5}{7}, \dfrac{7}{12}, \dfrac{9}{19}, \ldots\right\}$

$a_1 = \dfrac{2(1)+1}{1^2+3} = \dfrac{3}{4}$ $\qquad a_2 = \dfrac{2(2)+1}{2^2+3} = \dfrac{5}{7}$

$a_3 = \dfrac{2(3)+1}{3^2+3} = \dfrac{7}{12}$ $\qquad a_4 = \dfrac{2(4)+1}{4^2+3} = \dfrac{9}{19}$.

b) $\{b_n\} = \left\{ \frac{(-1)^n}{(n+1)!} \right\} = \left\{ -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, \frac{1}{120}, \dots \right\}.$

$b_1 = \frac{(-1)^1}{(1+1)!} = \frac{-1}{2!} = -\frac{1}{2}$; $b_2 = \frac{(-1)^2}{(2+1)!} = \frac{1}{3!} = \frac{1}{6}$,

$b_3 = \frac{(-1)^3}{(3+1)!} = \frac{-1}{4!} = -\frac{1}{24}$; $b_4 = \frac{(-1)^4}{(4+1)!} = \frac{1}{5!} = \frac{1}{120}$

✓

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c) $\{c_n\} = \{\cos(n\pi)\} = \{-1, 1, -1, 1 \dots\}$

$c_1 = \cos(\pi) = -1$, $c_2 = \cos(2\pi) = 1$

$c_3 = \cos(3\pi) = -1$, $c_4 = \cos(4\pi) = 1$

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d) $\underbrace{d_1 = 2}$; $d_2 = -1$; $\boxed{d_{n+2} = 2d_{n+1} + d_n - n!}$

$d_1 = 2$, $d_2 = -1$.

d_3 , $d_4 = ?$

$n=1$

$d_3 = d_{1+2} = 2d_2 + d_1 - 1! = 2(-1) + 2 - 1 = -1.$

$d_4 = d_{2+2} = 2d_3 + d_2 - 2! = 2(-1) + (-1) - 2 = -5$

$n=2$

$\{d_n\} = \{2, -1, -1, -5, \dots\}$

General formula of a sequence:

Ex: Put the following sequence into its general formula

a) $\sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots =$ $\left\{\sqrt{n+1}\right\}_{n=1}^{\infty}$ or $\left\{\sqrt{n}\right\}_{n=2}^{\infty}$ or $\left\{\sqrt{n+2}\right\}_{n=0}^{\infty}$

b) $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \ldots =$ $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$

c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots =$ $\left\{\frac{n-1}{n}\right\}_{n=2}^{\infty}$ or $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

d) $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots =$ $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=2}$ or $\left\{\frac{(-1)^{n}}{n+1}\right\}_{n=1}$

e) $\left\{-\frac{2}{3},\frac{3}{9},-\frac{4}{27},\frac{5}{81},\ldots,\frac{(-1)^n(n+1)}{3^n}\right\} =$ $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$

f) $\left\{\frac{3}{5},-\frac{4}{25},\frac{5}{125},-\frac{6}{625},\frac{7}{3125},\ldots\right\} =$ $\left\{\frac{(n+2)(-1)^{n+1}}{5^n}\right\}_{n=1}$

$5^1 \quad 5^2 \quad 5^3 \quad 5^4 \quad 5^5$

Ex: Recursive Formula: (Fibonacci Sequence)

$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, ...\}$

$a_1 = 1$, $a_2 = 1$

$a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

$\{a_n\} = \{a_1, a_2, a_3 \ldots$

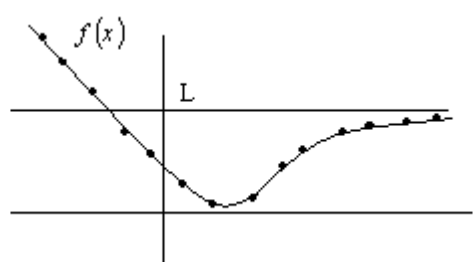
$\longrightarrow \longrightarrow \longrightarrow n \to \infty$

Limit of a sequence: where does a sequence go to? i.e. what is the number (only one) that a sequence will be eventually approaches to.

Def: A sequence $\{a_n\}$ has the limit L and we write $\boxed{\lim_{n\to\infty} a_n = L}$. If we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent)

A more precise version of limit:

A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ whenever n>N.



Def: Diverges to Infinity:

The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N such that for all n larger than N , $a_n > n$. If this condition holds we write $\lim_{n\to\infty} a_n = \infty$

$\pm\infty$, DNE.

Theorem 1: $\lim_{n\to\infty} a_n = A$; $\lim_{n\to\infty} b_n = B$

$a)$ $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$;

$b)$ $\lim_{n\to\infty}(a_n b_n) = AB$;

$c)$ $\lim_{n\to\infty}(ka_n) = kA$;

$d)$ $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$; $B \neq 0$

Squeeze Theorem : $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

L

Theorem 3: Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \to f(L)$

$f(a_1), f(a_2) \ldots, f(a_n)$

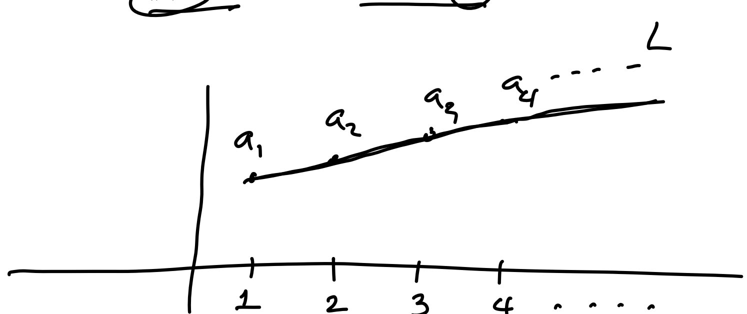
$\lim_{n\to\infty} f(a_n) = f(L)$

Theorem 4: Suppose that $f(x)$ is a function defined for all $x \ge x_0$ and that $\{a_n\}$ is a sequence of real numbers such that $f(n) = a_n$ for all $n \ge n_0$. Then

$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$

$f(x) = a_x$

Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$



Ex: Evaluate the limit of the following sequences:

a) $\{a_n\} = \left\{ \dfrac{2n^3 - 2n + 7}{\sqrt{9n^6 - 5n + 3}} \right\}$

dominant terms.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3 - 2n + 7}{\sqrt{9n^6 - 5n + 3}} \div \frac{n^3}{n^3} = \lim_{n\to\infty} \frac{2 - \frac{2}{n^2} + \frac{7}{n^3}}{\sqrt{9 - \frac{5}{n^5} + \frac{3}{n^6}}} = \frac{2}{\sqrt{9}} = \boxed{\frac{2}{3}}$$

$[e^3]^2 = e^6$

b) $\{b_n\} = \left\{ \left(1 + \dfrac{3}{n}\right)^{2n} \right\} \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{3}{n}\right)^{2n} = 1^\infty$

$$\ln b_n = \ln\left(1 + \frac{3}{n}\right)^{2n} = 2n \ln\left(1 + \frac{3}{n}\right).$$

$$\lim_{n\to\infty} \ln b_n = \lim_{n\to\infty} \left[2n \ln\left(1 + \frac{3}{n}\right)\right] = 2 \lim_{n\to\infty} \frac{\ln\left(1 + \frac{3}{n}\right)}{1/n} = \frac{0}{0}.$$

$$\overset{L'H}{=} 2 \lim_{n\to\infty} \frac{\frac{1}{1 + \frac{3}{n}} \left(-\frac{3}{n^2}\right)}{-\frac{1}{n^2}} = 2 \lim_{n\to\infty} \left(\frac{1}{1 + \frac{3}{n}}\right) = 3$$

$$\boxed{\lim_{n\to\infty} b_n = e^6}$$

c) $\{c_n\} = \left\{ \dfrac{\sin^2(2n+1)}{\sqrt{n^3 + 2}} \right\}$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\sin^2(2n+1)}{\sqrt{n^3 + 2}}.$$

$$0 \le \sin^2(2n+1) \le 1$$

$$0 \le \frac{\sin^2(2n+1)}{\sqrt{n^3 + 2}} \le \frac{1}{\sqrt{n^3 + 2}}.$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\sin^2(2n+1)}{\sqrt{n^3+2}} \le \lim_{n\to\infty} \frac{1}{\sqrt{n^3+2}} \to 0$$

$= 0$

By the Squeeze Thm,

$$\lim_{n\to\infty} c_n = 0.$$

d) $\{d_n\} = \{1 + (-1)^n\}$.

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \left[1 + (-1)^n\right].$$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} (-1)^n \quad \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

$$= 1 + \text{DNE} = \text{Divergent}$$

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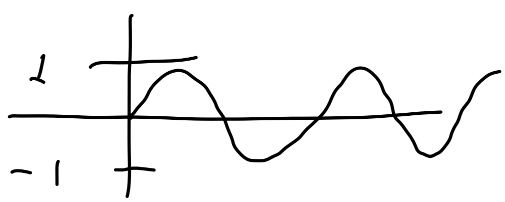
e) $\{e_n\} = \{\sin(n)\}$

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} \sin(n) = \text{DNE} = \text{Divergent}.$$



Note:

$$\lim_{n\to\infty} \#^n = \lim_{n\to\infty} r^n.$$

Constant

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{Divergent} & \text{if } |r| \ge 1 \end{cases}$$

f) $\{a_n\} = \left\{\dfrac{9^n + 7^n}{9^n + 5^n}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{9^n + 7^n}{9^n + 5^n} \div \frac{9^n}{9^n}.$$

$\frac{7}{9} < 1$

$$= \lim_{n\to\infty} \frac{1 + \left(\frac{7}{9}\right)^n \to 0}{1 + \left(\frac{5}{9}\right)^n \to 0} = \frac{1}{1} = \boxed{1}$$

f') $\{a_n\} = \left\{\dfrac{3\cdot 7^{n+1} + 5\cdot 8^n - 2}{8^{n+1} - 2\cdot 6^n + 4}\right\}.$

$$\lim a_n = \lim_{n\to\infty} \frac{3\cdot 7^{n+1} + 5\cdot 8^n - 2}{8^{n+1} - 2\cdot 6^n + 4} \div \frac{8^n}{8^n} = \lim_{n\to\infty} \frac{21\cdot\left(\frac{7}{8}\right)^n + 5 - 2\left(\frac{1}{8}\right)^n}{8 - 2\left(\frac{6}{8}\right)^n + 4\left(\frac{1}{8}\right)^n}$$

$\to 0$ $\to 0$

$$\lim_{n\to\infty} \quad 8^n \qquad = \boxed{\frac{5}{8}}$$

g) $\{a_n\} = \{\ln(n+2) - \ln(3n+2)\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\ln(n+2) - \ln(3n+2)\right] = \infty - \infty$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+2}{3n+2}\right) = \ln\left[\underbrace{\lim_{n\to\infty}\frac{n+2}{3n+2}}_{=\frac{1}{3}}\right] = \boxed{\ln\left(\frac{1}{3}\right)}$$

h) $\{a_n\} = \left\{\frac{n!}{n^n}\right\}$

Note: $n! = 1\cdot 2\cdot 3\cdot 4 \cdots n$.
$n^n = \underbrace{n\cdot n\cdot n\cdot n \cdots n}_{n\text{-time}}$.

$$0 \le a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} \le \frac{1\cdot \cancel{n}\cdot \cancel{n}\cdots \cancel{n}}{n\cdot \cancel{n}\cdot \cancel{n}\cdots \cancel{n}} = \frac{1}{n}.$$

$$0 \le a_n \le \frac{1}{n}.$$

$$\underbrace{\lim_{n\to 0} 0}_{=0} \le \lim_{n\to\infty} a_n \le \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}.$$

By the squeeze Thm!
$\lim_{n\to\infty} a_n = \lim \frac{n!}{n^n} = 0$

g) $\{a_n\} = \{\sqrt[n]{n}\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty}\left(n^{\frac{1}{n}}\right) = \infty^0$$

$$\ln a_n = \ln \sqrt[n]{n} = \ln n^{\frac{1}{n}} = \frac{1}{n}\ln n = \frac{\ln n}{n}.$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$$

$$\lim_{n\to\infty} \ln a_n = 0 \Rightarrow \underbrace{e^{\lim_{n\to\infty}\ln a_n}} = e^0$$

$$\lim_{n\to\infty} e^{\ln a_n} = \lim_{n\to\infty} a_n = \boxed{\lim_{n\to\infty}\sqrt[n]{n} = e^0 = 1.}$$

Note: For what values of r is the sequence $\{r^n\}$ convergence?

Sol: $\lim_{n\to\infty} r^n = \begin{cases} \infty \text{ if } r > 1 \\ 1 \text{ if r} = 1 \\ 0 \text{ if } 0 < \text{r} < 1 \end{cases}$ Demonstrate this by plotting point for n.

Theorem: The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if r} = 1 \\ 0 & \text{if -1} < \text{r} < 1 \end{cases}$$

Monotone and Bounded Sequences:

Def: Let $\{a_n\}$ be a sequence of real numbers.

- The sequence is monotone increasing if $a_n \le a_{n+1}$ for all $n \ge 1$
- The sequence is monotone decreasing if $a_n \ge a_{n+1}$ for all $n \ge 1$
- The sequence is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$
- The sequence is bounded below if there is a number m such that $a_n \ge M$ for all $n \ge 1$

(If a sequence is bounded above and bounded below, we say that the sequence is bounded. If a sequence is not bounded, we say that it is unbounded.)

Ex: a) For positive integer n, let $a_n = \sqrt{n^4 + n^3} - n^2$. Show that the sequence $\{a_n\}$ is monotone increasing and unbounded.

b) Let $\{a_n\} = \left\{ \frac{(-1)^n}{n} \right\}$. Show that the sequence $\{a_n\}$ is bounded but not monotone.

c) Show that the sequence $a_n = \frac{3}{n+5}$ is monotone decreasing.

d) Show that the sequence $a_n = \frac{n}{n^2+1}$ is monotone decreasing.

c) $\{a_n\} = \left\{\dfrac{3}{n+5}\right\}$ ➔ it's monotone decreasing.

d) $\{a_n\} = \left\{\dfrac{n}{n^2+1}\right\}$

<u>The Monotone Sequence Theorem</u>: Let $\{a_n\}$ be a monotone increasing sequence of real numbers.

a) if $\{a_n\}$ is bounded above, then $\lim_{n\to\infty} a_n$ exists.

b) if $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$

Ex: Define a sequence $\{a_n\}$ by the recursion relationship $a_1 = 1$; $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that the sequence converges and find its limit.

Ex: Investigate the sequence $\{a_n\}$ defined by the recursive definition

$a_1 = 2; a_{n+1} = \frac{1}{2}(a_n + 6)$ for n = 1, 2, 3,…

Should Know.

1.	$\lim_{n\to\infty} \frac{\ln n}{n} = 0$	2.	$\lim_{n\to\infty} \sqrt[n]{n} = 1$ ✓ $n^{\frac{1}{n}}$	3.	$\lim_{n\to\infty} x^{1/n} = 1$ (x>0) ✓
4.	$\lim_{n\to\infty} x^n = 0$ $(\lvert x\rvert < 1)$	5.	$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ (any x)	6.	$\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x) ✓

Ex: Evaluate the following limits:

✓ a) $\lim_{n\to\infty} \frac{\ln\left(n^3\right)}{4n} = \lim_{n\to\infty} \frac{3\ln n}{4n} = \frac{3}{4}\underbrace{\lim_{n\to\infty} \frac{\ln n}{n}}_{=0} = \boxed{0}$

b) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} (n^2)^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^2 = \left[\lim_{n\to\infty} n^{\frac{1}{n}}\right]^2 = [1]^2 = 1.$

c) $\lim_{n\to\infty}\left(\frac{3^n}{n^3}\right) = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{3n^2}$

$= \frac{\ln 3}{3} \lim_{n\to\infty} \frac{3^n}{n^2} = \frac{\infty}{\infty} \overset{L'H}{=} \frac{\ln 3}{3} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{2n}.$

$= \frac{(\ln 3)^2}{6} \lim_{n\to\infty} \frac{3^n}{n} = \frac{\infty}{\infty} \overset{L'H}{=} \frac{(\ln 3)^2}{6} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{1}.$

$= \frac{(\ln 3)^3}{6} \lim_{n\to\infty} 3^n \to \infty = \infty$ Divergent

d) $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} =$

e) $\lim_{n\to\infty}(n+4)^{1/(n+4)}$ ✓ let $u=n+4$, $n\to\infty \Rightarrow u\to\infty$

$\sqrt[n]{n}$.

$\lim_{u\to\infty} u^{\frac{1}{u}} = \boxed{1}$

f) $\lim_{n\to\infty}\frac{(10/11)^n}{(9/10)^n+(11/12)^n} \div \frac{\left(\frac{11}{12}\right)^n}{\left(\frac{11}{12}\right)^n}$

$= \lim_{n\to\infty} \frac{\left(\frac{120}{121}\right)^{n}\to 0}{\left(\frac{108}{110}\right)^{n}\to 0 + 1} = \frac{0}{0+1} = \frac{0}{1} = \boxed{0}$

g) $\lim_{n\to\infty}\frac{3^n6^n}{2^{-n}n!} = \lim_{n\to\infty}\frac{3^n\cdot 6^n\cdot 2^n}{n!}$

$= \lim_{n\to\infty}\frac{36^n}{n!} = 0$ $\left\{ \text{b/c } \lim_{n\to\infty}\frac{x^n}{n!} = 0 \text{ for any } x. \right.$

h) $\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{3n}\right)^n}{\left(1-\frac{1}{3n}\right)^n} = \frac{e^{1/3}}{e^{-1/3}}$

$= \boxed{e^{2/3}}$