Series

Def: Series is the sum of all elements of a sequence.

i.e. Given a sequence $\{a_n\} = \{a_1, a_2, a_3, a_4 \cdots\}$

$$\text{Series} = a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{i=1}^{\infty} a_i = \sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} a_n$$

Find the limit of a Series: $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots = S$

Sequence of ***Partial Sum***: Given a series: $\sum_{n=1}^{\infty} a_n \overset{?}{=} S$.

$$S = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

$\vdots$

partial Sum sequence: $\{S_1, S_2, S_3, S_4, \cdots\}$

$$\boxed{S = \lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n}$$

Def: Given a series $\sum_{n=1}^{\infty} a_n$, let $S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$ be the n^th^ partial sum of the series. If the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$\sum_{n=1}^{\infty} a_n = S$, the number S is called the sum of the series. Otherwise, the series is called divergent.

HQH

$a_1 = ar^0,\ a_2 = a\cdot r^1,\ a_3 = a\cdot r^2,\ a_4 = a\cdot r^3$

$\{a_n\} = \{ar^0, ar^1, ar^2, ar^3, ar^4, \dots\}$. $\quad a_n = a\cdot r^{n-1}$

Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} S_n = S$.

$S_n = a_1 + a_2 + a_3 + \dots + a_n$.

$S_n = a + ar + ar^2 + \dots + ar^{n-1}$

multiply by r

$$\begin{aligned} S_n &= a + ar + ar^2 + \dots + ar^{n-1} \\ rS_n &= ar + ar^2 + ar^3 + \dots + ar^n \end{aligned}$$

$S_n - rS_n = a - ar^n$.

$S_n(1-r) = a(1-r^n) \Rightarrow S_n = \dfrac{a(1-r^n)}{1-r}$.

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a(1-\lim_{n\to\infty} r^n)}{1-r}$ $\quad \lim_{n\to\infty} r^n =$

where $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r|\geq 1 \end{cases}$

$$\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r|\geq 1 \end{cases}}$$

Ex: Evaluate the following:

a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} a\cdot r^{n-1}$ $\begin{cases} a = 3 \\ r = \left|\frac{1}{2}\right| < 1 \end{cases}$

$\Rightarrow$ By G.S. $\Rightarrow$ Convergent $\Rightarrow$ Limit $= \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{2}} = \boxed{6}$

Note: Geometric Series.

$$\sum_{n=1}^{\infty} a\cdot r^{n-1} = \sum_{n=0}^{\infty} a\cdot r^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r|\geq 1 \end{cases}$$

b) $\dfrac{1}{2^0} - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dots + \left(-\dfrac{1}{2}\right)^{n-1} + \dots =$ $\quad (4 = 2^2,\ 8 = 2^3)$

$= \sum_{n=1}^{\infty} 1\cdot\left(-\frac{1}{2}\right)^{n-1}$ $\begin{cases} a = 1 \\ r = \left|-\frac{1}{2}\right| < 1 \end{cases}$ $\Rightarrow$ Convergent by G.S.

$\Rightarrow$ Limit $=$ Sum $= \dfrac{a}{1-r} = \dfrac{1}{1+\frac{1}{2}} = \boxed{\frac{2}{3}}$

Series of $a(\text{constant})^{\text{power } n}$

$\Rightarrow$ Geometric Series.

c) $\dfrac{\pi}{2} + \dfrac{\pi^2}{4} + \dfrac{\pi^3}{8} + \dots = \sum_{n=1}^{\infty} \left(\dfrac{\pi}{2}\right)^{n-1} \Rightarrow \begin{cases} a = 1 \\ r = \frac{\pi}{2} \end{cases}$

$|r| = \left|\frac{\pi}{2}\right| > 1$

$\Rightarrow$ Divergent by G.S.

d) $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \dots = \sum\ ?$ $\quad (5\cdot 1,\ 5\cdot 2,\ 5\cdot 4,\ 5\cdot 8;\ 3^0)$

$= \dfrac{5\cdot 2^0}{3^0} - \dfrac{5\cdot 2}{3^1} + \dfrac{5\cdot 2^2}{3^2} - \dfrac{5\cdot 2^3}{3^3} + \dfrac{5\cdot 2^4}{3^4} - \dots = \sum_{n=1}^{\infty} 5\cdot\left(-\frac{2}{3}\right)^{n-1}$

$\begin{cases} a = 5 \\ r = \left|-\frac{2}{3}\right| < 1 \end{cases}$ $\Rightarrow$ Convergent by G.S.

$\Rightarrow$ Limit $= \frac{a}{1-r} = \frac{\;}{1+\frac{2}{3}} = \boxed{\;}$

e) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+2}} = \sum_{n=1}^{\infty} a\, r^{n-1}$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n-1}\cdot 4^3} = \sum_{n=1}^{\infty} \frac{1}{64}\left(\frac{3}{4}\right)^{n-1} \quad \begin{cases} a = \frac{1}{64} \\ r = \left|\frac{3}{4}\right| < 1 \end{cases}$$

$\Rightarrow$ Convergent by G.S.

$$\Rightarrow \text{Limit} = \frac{a}{1-r} = \frac{1/64}{1-\frac{3}{4}} = \frac{1}{64-48} = \boxed{\frac{1}{16}}$$

f) $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{2n+1}} = \sum_{n=0}^{\infty} a\cdot r^n$

$$= \sum_{n=0}^{\infty} \frac{3^n\cdot 3^{-1}}{4^n\cdot 2} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{3}{4}\right)^n \quad \begin{cases} a = \frac{1}{6} \\ r = \left|\frac{3}{4}\right| < 1 \end{cases}$$

$\Rightarrow$ Convergent by G.S. $\Rightarrow$ Limit $S = \frac{a}{1-r}$

$$S = \frac{1/6}{1-3/4} = \frac{2}{12-9} = \boxed{\frac{2}{3}}$$

g) $\sum_{n=0}^{\infty} \frac{2^{3n-1}}{7^{n+1}} = \sum_{n=0}^{\infty} a\cdot r^n$

$$= \sum_{n=0}^{\infty} \frac{8^n\cdot 2^{-1}}{7^n\cdot 7} = \sum_{n=0}^{\infty} \frac{1}{14}\left(\frac{8}{7}\right)^n \quad \begin{cases} a = \frac{1}{14} \\ r = \left|\frac{8}{7}\right| > 1 \end{cases}$$

$$2^{2n+5} = 2^{2n}\cdot 2^5 = 4^n\cdot 2^5 = 4^{n-1}\cdot 4\cdot 2^5$$

2^2

$\Rightarrow$ it's divergent by G.S.

Ex: $$\sum_{n=3} \frac{2^{2n+1}}{5^{n+3}} = \sum_{n=1}^{\infty} \frac{2^{2(n+2)+1}}{5^{n+2+3}} = \sum_{n=1}^{\infty} \frac{2^{2n+5}}{5^{n+5}} = \sum_{n=1}^{\infty} a\cdot r^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{4^{n-1}\cdot 2^7}{5^{n-1}\cdot 5^6} = \sum_{n=1}^{\infty} \frac{2^7}{5^6}\left(\frac{4}{5}\right)^{n-1} \quad \begin{cases} a = \frac{2^7}{5^6} \\ r = \left|\frac{4}{5}\right| < 1 \end{cases}$$

$\Rightarrow$ It's convergent by G.S. $\Rightarrow$ Limit $S = \frac{a}{1-r} = \frac{2^7/5^6}{1-\frac{4}{5}} = \#$

Ex: Determine equivalent fraction:

$5.\overline{23} = 5.232323\ldots$

$= 5 + 0.23 + 0.0023 + 0.000023 + \cdots$

$= 5 + \underbrace{\frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \frac{23}{10^8} + \cdots}$

$= 5 + \sum_{n=1}^{\infty} \frac{23}{10^{2n}} \rightsquigarrow \sum a \cdot r^{n-1}$

$\sum_{n=1}^{\infty} \frac{23}{100^{n-1} \cdot 100} = \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1}$ $\begin{cases} a = \frac{23}{100} \\ r = \left|\frac{1}{100}\right| < 1 \end{cases}$

Limit $= \frac{a}{1-a} = \frac{23/100}{1 - 1/100} = \frac{23}{99}$.

$5.\overline{23} = 5 + \frac{23}{99} = \frac{5(99)+23}{99} = \boxed{\frac{518}{99}}$

$$\boxed{\text{Geometric Series: } S = \sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=0}^{\infty} a \cdot r^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}}$$

Def: A telescoping series is one in which the n^{th} term can be expressed in the form $a_n = b_n - b_{n+1}$

$(\quad)^n$

Convergence of a telescoping series:

If $\sum_{n=1}^{\infty} a_n$ is a telescoping series with $a_n = b_n - b_{n+1}$ then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $\{b_n\}$ converges. Furthermore, if $\{b_n\}$ converges to L, then $\sum_{n=1}^{\infty} a_n$ converges to L

Not (#)?

Ex: Find the sum of the following:

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12} = \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)} = \sum_{n=1}^{\infty} \left(\frac{A}{n+3} + \frac{B}{n+4}\right)$

$A\Big|_{n=-3} = 1 \quad ; \quad B\Big|_{n=-4} = -1$

$\sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n+3} - \frac{1}{n+4}\right)}_{a_n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{n+4}\right) = \boxed{\frac{1}{4}}$

where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

$= \underbrace{\left(\frac{1}{4} - \frac{1}{5}\right)}_{n=1} + \underbrace{\left(\frac{1}{5} - \frac{1}{6}\right)}_{n=2} + \underbrace{\left(\frac{1}{6} - \frac{1}{7}\right)}_{n=3} + \cdots + \underbrace{\left(\frac{1}{n+3} - \frac{1}{n+4}\right)}_{n=n}$

$= \frac{1}{4} - \frac{1}{n+4}$

b) $\sum_{n=0}^{\infty} \underbrace{\left(e^{1/(n+3)} - e^{1/(n+2)}\right)}_{a_n} = \lim_{n\to\infty} S_n$

where $S_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n$

$S_n = \underbrace{\left(e^{\frac{1}{3}} - e^{\frac{1}{2}}\right)}_{n=0} + \underbrace{\left(e^{\frac{1}{4}} - e^{\frac{1}{3}}\right)}_{n=1} + \underbrace{\left(e^{\frac{1}{5}} - e^{\frac{1}{4}}\right)}_{n=2} + \underbrace{\left(e^{\frac{1}{6}} - e^{\frac{1}{5}}\right)}_{n=3} + \cdots + \underbrace{\left(e^{\frac{1}{n+3}} - e^{\frac{1}{n+2}}\right)}_{n=n}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-e^{\frac{1}{2}} + e^{\frac{1}{n+3}}\right) = -e^{\frac{1}{2}} + e^{0}$

$= \boxed{1 - \sqrt{e}}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3} = \sum_{n=1}^{\infty}\frac{1}{(n+3)(n+1)} = \sum_{n=1}^{\infty}\left(\frac{A}{n+1}+\frac{B}{n+3}\right)$

$$A\Big|_{n=-1} = \frac{1}{-1+3} = \frac{1}{2} \quad ; \quad B\Big|_{n=-3} = \frac{1}{-3+1} = -\frac{1}{2}.$$

$$\sum_{n=1}^{\infty}\left(\frac{1/2}{n+1}-\frac{1/2}{n+3}\right) = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)} = \frac{1}{2}\lim_{n\to\infty} S_n.$$

where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$$= \underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{n=1} + \underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{n=2} + \underbrace{\left(\frac{1}{4}-\frac{1}{6}\right)}_{n=3} + \underbrace{\left(\frac{1}{5}-\frac{1}{7}\right)}_{n=4} + \underbrace{\left(\frac{1}{6}-\frac{1}{8}\right)}_{n=5}$$

$$+ \cdots + \underbrace{\left(\frac{1}{n+1}-\frac{1}{n+3}\right)}_{n=n}.$$

$$S = \frac{1}{2}\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \boxed{\frac{5}{12}}$$

G.S.
T.S.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Divergent Test Theorem (D.T.T.)

The test for Divergence: If $\boxed{\lim_{n\to\infty} a_n}$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then $\boxed{\sum_{n=1}^{\infty} a_n \text{ is divergent.}}$ D.T.T.

Ex: Show that the following series is divergent.

a) $\displaystyle\sum_{n=1}^{\infty}\cos(n\pi) \overset{\text{let}}{=\!=} a_n.$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\cos(n\pi) = \text{DNE} \neq 0.$$

$\therefore$ by D.T.T. $\sum_{n=1}^{\infty}\cos(n\pi)$ is divergent

$$\lim_{n\to\infty} a_n = \begin{Bmatrix}\text{DNE} \\ \neq 0\end{Bmatrix} \Rightarrow \sum a_n \text{ is divergent}$$

Note: If $\lim_{n\to\infty} a_n = 0$

$\Rightarrow$ Inconclusive

(use wrong technique).

b) $\displaystyle\sum_{n=1}^{\infty}\frac{5n^3+2n-7}{2n^3+1} \overset{\text{let}}{=\!=} a_n.$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5n^3+2n-7}{2n^3+1} = \frac{5}{2} \neq 0$$

$\Rightarrow \sum a_n$ is divergent by D.T.T.

c) $\displaystyle\sum^{\infty} e^{\frac{5}{n^2+2}} \overset{\text{let}}{=\!=} a_n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\left(\frac{5}{n^2+2}\right)\to 0} = e^0 = 1 \neq 0$

$n=0$ $\therefore$ by D.T.T. $\Rightarrow \sum a_n$ is divergent.

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series

$$\sum_{n=1}^{\infty} ca_n \quad \text{and} \quad \sum_{n=1}^{\infty} (a_n \pm b_n)$$

Ex: Find the sum of the series

a) $$\sum_{n=1}^{\infty}\left(\frac{1}{3^{n-1}}+\frac{1}{n(n+1)}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}}_{G.S.} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{T.S.}$$

$$= \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}} + \sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)$$

$$A\Big|_{n=0} = \frac{1}{0+1} = 1$$

$$B\Big|_{n=-1} = \frac{1}{-1} = -1$$

$a=1$

$r=\left|\frac{1}{3}\right|<1$

$$= \frac{a}{1-r} = \frac{1}{1-1/3} = \boxed{\frac{3}{2}}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$= \underbrace{\left(1-\frac{1}{2}\right)}_{n=1} + \underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=2} + \underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=3} + \cdots + \underbrace{\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{n=n}$$

$$= 1 - \frac{1}{n+1}$$

$$S = \frac{3}{2} + 1 = \boxed{\frac{5}{2}}$$

b) $$\sum_{n=0}^{\infty} \frac{\cos(n\pi)+2^{n+1}}{5^n}$$

$$= \sum_{n=0}^{\infty} \frac{\overbrace{\cos(n\pi)}^{=(-1)^n}}{5^n} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$$

Note: $\cos(n\pi) = \underset{n=0}{\cos 0} = 1,\ \underset{n=1}{\cos\pi} = -1,\ \underset{n=2}{\cos(2\pi)} = 1,\ \underset{n=3}{\cos(3\pi)} = -1$

$$\cos(n\pi) = (-1)^n$$

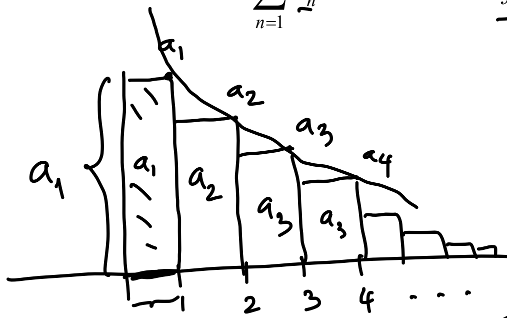
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \sum_{n=0}^{\infty} \frac{2^n \cdot 2}{5^n} = \sum_{n=0}^{\infty} 1\left(-\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} 2\left(\frac{2}{5}\right)^n$$

$$= \frac{1}{1+\frac{1}{5}} + \frac{2}{1-\frac{2}{5}} = \frac{5}{6} + \frac{10}{3}$$

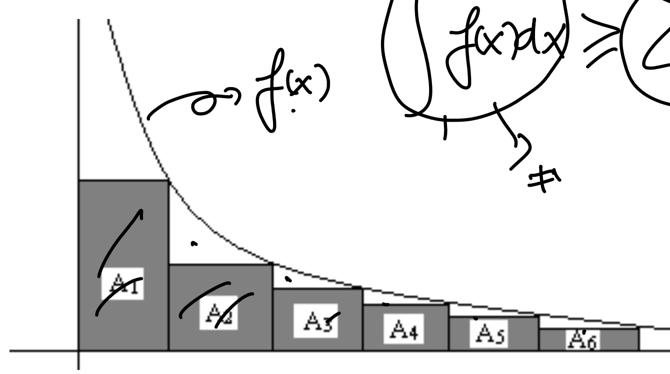
$= \frac{5+20}{6} = \boxed{\frac{25}{6}}$

11.3 *The Integral Test and Estimates of Sums*

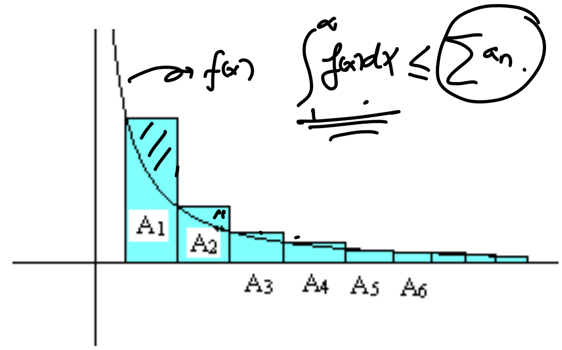
Given a series $\sum_{n=1}^{\infty} a_n$ and consider $f(x) = a_x$ (*i.e. replaced n by x*)



$\int^{\infty} f(x)dx \geq \sum a_n$



$\int^{\infty} f(x)dx \leq \sum a_n$



Case I

Case II

Introducing the test be given a series $\sum_{n=1}^{\infty} a_n$ and define a function $f(n) = a_n$

Case I: $\sum_{n=1}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \rightarrow$ If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Case II: $\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)dx \rightarrow \int_1^{\infty} f(x)dx$ is divergent, and then $\sum_{n=1}^{\infty} a_n$ is divergent.

(I.T.T.)

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Suppose that $a_n = f(n)$, $a_x = f(x)$ where f is continuous, positive, decreasing function of x for all $x \geq N$ for some positive integer N. Then the series $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ both converge or both diverge.

I.T.T. Given $\sum a_n \Longrightarrow$ define $f(x) = a_x$ {Replace n by x.

$f(x)$ is {1. continuous 2. positive 3. decreasing} for $x \geq 1$.

If $\int^{\infty} f(x)dx$ is convergent $\Longrightarrow \sum a_n$ is convergent

is divergent

is divergent $\Rightarrow$

Review p – test theorem: for Improper Integrals.

$$\int_{a>0}^{\infty} \frac{1}{x^p}dx \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1. \end{cases}$$

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ let $a_n \Rightarrow$ define $f(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$ $\begin{cases} 1.\ \text{continuous } x>0 \\ 2.\ \text{positive } x>0 \\ 3.\ \text{decreasing } x\ge 1 \end{cases}$

Consider $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^{3/2}}dx$ $\left\{ p=\frac{3}{2}>1 \text{ is convergent by } p\text{-Test} \right.$

By I.T.T. $\Rightarrow \sum a_n$ is convergent.

b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ let $a_n \Rightarrow$ define $f(x) = \frac{1}{x\sqrt{\ln x}}$ $\left.\begin{cases} 1.\ \text{cont.} \\ 2.\ \text{positive} \\ 3.\ \text{decreasing} \end{cases}\right\}$ for $x\ge 2$.

Consider $\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$ $\left\{ \begin{array}{l} \text{let } u=\ln x \ \begin{cases} x=\infty \Rightarrow u=\infty \\ x=2 \Rightarrow u=\ln 2 \end{cases} \\ du = \frac{1}{x}dx \end{array}\right.$

$= \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = \int_{\ln 2}^{\infty} \frac{du}{u^{1/2}}$ $\left\{ p=\frac{1}{2}<1 \text{ is divergent by } p\text{-Test} \right.$ $\Rightarrow$ By I.T.T. $\sum a_n$ is divergent.

P – test for series $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ convergent}}$

Sol: Let $f(x) = \frac{1}{x^p} \Rightarrow \int_1^{\infty} \frac{1}{x^p}dx$ is convergent if $p>1$ and it's divergent if $p\le 1$

$$\boxed{\text{The p-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \underline{\text{convergent}} \text{ if } \underline{p>1} \text{ and it's } \underline{\text{divergent}} \text{ if } \underline{p\le 1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } \underline{p\le 1} \end{cases}$$

G.S. $\boxed{\sum a\cdot r^n : \begin{array}{l} |r|<1 \Rightarrow \text{convergent} \\ |r|\ge 1 \Rightarrow \text{divergent} \end{array}}$ $\frac{a}{1-r}$

$$s = \lim S_n$$

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{1}{(5n)^3} = \sum_{n=1}^{\infty} \frac{1}{125 \cdot n^3} = \frac{1}{125} \sum_{n=1}^{\infty} \frac{1}{n^3}$ $\left\{ \begin{array}{l} p = 3 > 1 \Rightarrow \\ \text{it's convergent} \\ \text{by } p\text{-Test}. \end{array} \right.$

b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{2}{n^{3/4}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ $\left\{ \begin{array}{l} p = \frac{3}{4} < 1 \\ \Rightarrow \text{it's divergent} \\ \text{by } p\text{-Test}. \end{array} \right.$

$$\left\{ \begin{array}{l} 1.\ \text{G.S} \\ 2.\ \text{T.S.} \\ 3.\ \text{D.T.T.} \\ 4.\ \text{I.T.T.} \\ 5.\ p\text{-Test} \end{array} \right\}$$

c) $\sum_{n=1}^{\infty} \frac{\ln n}{n} \rightarrow f(x) = \frac{\ln x}{x}$ $\left\{ \begin{array}{l} 1.\ \text{cont.} \\ 2.\ \text{positive.} \\ 3.\ \text{decreasing.} \end{array} \right\}$ over $[1, \infty)$

$$\Rightarrow \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\ln x}{x}\,dx \quad \left\{ \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array} \right.$$

$$= \int_1^{\infty} u\,du = \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$$

$$= \frac{1}{2} \lim_{t \to \infty} (\ln x)^2 \Big|_1^t = \frac{1}{2} \lim_{t \to \infty} \left[(\ln t)^2 - 0 \right]$$

$$= \infty.$$

$\Rightarrow \sum a_n$ is divergent by I.T.T.

Estimate the sum of a series:

Given $\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{= S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{= R_n}$

$$S = \sum_{n=1}^{\infty} a_n = S_n + R_n$$

S_n : Approximation of the sum

$\boxed{R_n : \text{Error}}$ ← "small"

$$R_n = S - S_n$$

Remainder Estimate for the integral test:

Suppose $f(n) = a_n$, where f is continuous, positive, decreasing function for $x \ge n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)dx \le R_n \le \boxed{\int_{n}^{\infty} f(x)dx}$ Error

Error $\boxed{|R_n| < \int_n^{\infty} f(x)dx}$

Ex: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms.

$\boxed{\text{Estimate the error}}$ involved in this approximation.

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx S_{10} = a_1 + a_2 + a_3 + \cdots + a_{10}$$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} = \#.$$

Error : $|R_{10}| \le \int_{10}^{\infty} f(x)dx$ where $f(x) = a_x = \frac{1}{x^3}$.

$$\int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{t\to\infty} \int_{10}^{t} x^{-3} dx$$

$$= \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_{10}^{t} = -\frac{1}{2} \lim_{t\to\infty} \left[\frac{1}{t^2} - \frac{1}{100}\right]$$

$$= -\frac{1}{2}\left(-\frac{1}{100}\right) = \frac{1}{200} = 0.005 = 0.5\%$$

$n = ?$ $\qquad |R_n| < 0.0005.$

b) How many terms are required to ensure that the sum is accurate to within 0.0005?

$n = ?$ such that $|R_n| < 0.0005$

where $|R_n| < \underbrace{\int_n^\infty f(x)\,dx}_{n} < 0.0005.$

$$\Rightarrow \lim_{t\to\infty}\int_n^t \frac{1}{x^3}\,dx = \lim_{t\to\infty}\frac{x^{-2}}{-2}\Big|_n^t = -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{t^2} - \frac{1}{n^2}\right].$$

$\frac{1}{t^2} \to 0$

$$|R_n| < \frac{1}{2n^2} < 0.0005 \Rightarrow \frac{1}{0.0005} < 2n^2$$

$$n > \sqrt{\frac{1}{(0.0005)(2)}} = \sqrt{\frac{1}{0.001}} = \sqrt{1000} = 31.6$$

$n = ?$ $\qquad \boxed{n \geq 32 \text{ terms}}$

Ex: Determine how many terms are needed to ensure the error within 0.005.

$$\sum_{n=1}^{\infty}\frac{1}{n\left[\ln(3n)\right]^3}$$

$n = ?$ so that $|R_n| < 0.005.$

$$\Rightarrow |R_n| < \underbrace{\int_n^\infty f(x)\,dx}_{} < 0.005$$

$$\int_n^\infty \frac{1}{x\,[\ln(3x)]^3}\,dx \quad \begin{cases} \text{let } u = \ln(3x) \\ du = \frac{1}{3x}\cdot 3\,dx = \frac{1}{x}\,dx \end{cases}$$

$$\Rightarrow \int \frac{1}{u^3}\,du = \int u^{-3}\,du = \frac{u^{-2}}{-2} = -\frac{1}{2}\left[\frac{1}{(\ln(3x))^2}\right].$$

$$\Rightarrow -\frac{1}{2}\lim_{t\to\infty}\frac{1}{(\ln(3x))^2}\Big|_n^t$$

$$= -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{[\ln(3t)]^2} - \frac{1}{(\ln(3n))^2}\right].$$

$\to 0$.

$$|R_n| < \frac{1}{2[\ln(3n)]^2} < 0.005.$$

Trial & error $\Rightarrow$ $n = 10 \Rightarrow \dfrac{1}{2[\ln(30)]^2} = 0.04$

$n = 100 \Rightarrow \dfrac{1}{2[\ln(300)]^2} = 0.015 > 0.005$

$n = 10{,}000 \Rightarrow \dfrac{1}{2[\ln(30{,}000)]^2} = 0.0047 < 0.005$

$$\boxed{n \geq 10{,}000}$$

Q1H

Ex: How many terms are needed to ensure the error < 0.0005

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+2)^7}$$

Sol: Error $= |R_n| < \int_n^{\infty} f(x)\,dx$

where $f(x) = \dfrac{x}{(x^2+2)^7}$

$$\Rightarrow \int_n^{\infty} \frac{x}{(x^2+2)^7}\,dx = \lim_{t\to\infty}\int_n^{t} \frac{x}{(x^2+2)^7}\,dx$$

let $u = x^2+2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$.

$$= \lim_{t\to\infty}\int \frac{\frac{1}{2}du}{u^7} = \frac{1}{2}\lim_{t\to\infty}\int u^{-7}\,du = \frac{1}{2}\lim_{t\to\infty}\frac{u^{-6}}{-6}.$$

$$= -\frac{1}{12}\lim_{t\to\infty}\left(\frac{1}{u^6}\right) = -\frac{1}{12}\lim_{t\to\infty}\frac{1}{(x^2+2)^6}\Bigg|_n^{t}.$$

$$= -\frac{1}{12}\lim_{t\to\infty}\left[\underbrace{\frac{1}{(t^2+2)^6}}_{\to 0} - \frac{1}{(n^2+2)^6}\right].$$

$$= \frac{1}{12(n^2+2)^6} \le 0.0005.$$

Trial & error $\Rightarrow$ pick $n=5$

?

$$\frac{1}{12(5^2+2)^6} \le 0.0005.$$

||

-10

$2.15 \times 10 < 0.000$[illegible]