Direct Comparison Test Thm **The Comparison Tests**

(D.C.T.T.)

The Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

$0 \le$ small $\le$ big.

If $\sum$ big is convergent $\Rightarrow \sum$ small is convergent

$\sum$ small is divergent $\Rightarrow \sum$ big is divergent

a) $$\sum_{n=1}^{\infty} \frac{7n^2+5n^3+4\cdot 1}{\sqrt{4n^{12}+5n+4}} \le \sum_{n=1}^{\infty} \frac{7n^3+5n^3+4n^3}{\sqrt{n^{12}}} = \sum_{n=1}^{\infty} \frac{16n^3}{n^6} = 16\sum_{n=1}^{\infty}\frac{1}{n^3}$$ $p=3>1$ is convergent by p-Test.

Dominant terms: $\frac{n^3}{\sqrt{n^{12}}} = \frac{n^3}{n^6} = \frac{1}{n^{3>1}}$ ← convergent ⇒ Construct bigger $\sum$

∴ by DCTT. $\Rightarrow \sum a_n$ is convergent.

b) $$\sum_{n=1}^{\infty} \frac{\cancel{4}n^3+\cancel{5n}+\cancel{8}}{\sqrt{n^8+5n+12\cdot 1}} \ge \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^8+5n^8+12n^8}} = \sum_{n=1}^{\infty}\frac{n^3}{\sqrt{18n^8}} = \underbrace{\frac{1}{\sqrt{18}}\sum_{n=1}^{\infty}\frac{n^3}{n^4} = \frac{1}{\sqrt{18}}\sum_{n=1}^{\infty}\frac{1}{n}}_{p=1 \Rightarrow \text{divergent by p-Test.}}$$

Dominant terms: $\frac{n^3}{\sqrt{n^8}} = \frac{n^3}{n^4} = \frac{1}{n}$ ← Divergent ⇒ Construct smaller.

∴ by D.C.T.T. $\sum a_n$ is divergent.

$-1 \le \sin(\theta) \le 1 \le n^2$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+\sin(7n+\pi)}{n^5+\cancel{3n}+\cancel{5}} \leq \sum_{n=1}^{\infty}\frac{n^2+n^2}{n^5} = \sum_{n=1}^{\infty}\frac{2n^2}{n^5} = 2\sum_{n=1}^{\infty}\frac{1}{n^3}$ $\left\{\begin{array}{l} p=3>1 \\ \text{is convergent} \\ \text{by } p-\text{Test}\end{array}\right.$

≥ 1

Dominant terms: $\dfrac{n^2}{n^5} = \dfrac{1}{n^{3>1}}$ ← convergent ⇒ Construct bigger $\sum$.

∴ By D.C.T.T. ⇒ $\sum a_n$ is convergent.

$\displaystyle\sum_{n=1}^{\infty}\frac{n^2+4^n+4\cdot 1}{n^5\cdot 4^{n+1}} \leq \sum_{n=1}^{\infty}\frac{4^n+4^n+4\cdot 4^n}{n^5\cdot 4^{n+1}} = \sum_{n=1}^{\infty}\frac{6\cdot 4^n}{n^5\cdot 4^n\cdot 4} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}$ $\left\{\begin{array}{l} p=5>1 \text{ is} \\ \text{convergent} \\ \text{by } p-\text{Test.}\end{array}\right.$

Dominant terms: $\dfrac{4^n}{n^5\cdot 4^{n+1}} = \dfrac{4^n}{n^5\cdot 4^n\cdot 4} = \dfrac{1}{4n^5}$ → convergent ⇒ Construct bigger $\sum$

∴ by D.C.T.T. ⇒ $\sum a_n$ is also convergent.

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e) $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+\cancel{3\cos^2(n)}+\cancel{4}}{\sqrt{n^5+5n+7\cdot 1}} \geq \sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n^5+5n^5+7n^5}} = \sum_{n=1}^{\infty}\frac{n^2}{\sqrt{13n^5}} = \frac{1}{\sqrt{13}}\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n^5}}$

Dominant terms: $\dfrac{n^2}{\sqrt{n^5}} = \dfrac{n^2}{n^{5/2}} = \dfrac{1}{n^{\frac{5}{2}-2}} = \dfrac{1}{n^{1/2<1}}$ ⇒ divergent ⇒ Construct smaller.

$\displaystyle = \frac{1}{\sqrt{13}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{2}-2}} = \frac{1}{\sqrt{13}}\sum_{n=1}^{\infty}\frac{1}{n^{1/2<1}}$

⇒ divergent by $p-$Test

∴ by DCTT ⇒ $\displaystyle\sum_{n=1}^{\infty} a_n$ is also divergent

The Limit Comparison Test: (L.C.T.T.)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty}\frac{a_n}{b_n}=c$ where c is a ***finite number*** and $c>0$, then either both series converge or both diverge.
Suppose $a_n > 0$ *and* $b_n > 0$ for all $n \ge N$

1. $\lim_{n\to\infty}\frac{a_n}{b_n}=c>0 \Rightarrow$ both converge or both diverge.
2. $\lim_{n\to\infty}\frac{a_n}{b_n}=0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges*
3. $\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges*

Use L.C.T.T. when the series consistent of both + & − signs.

Ex: Test for convergence

a) $\sum_{n=1}^{\infty}\frac{7n^3-n+4}{\sqrt{49n^{10}-9n+4}}$ let a_n.

Let b_n = dominant terms of $a_n = \frac{n^3}{\sqrt{n^{10}}} = \frac{n^3}{n^5} = \frac{1}{n^2}$

Consider $c = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{7n^3-n+4}{\sqrt{49n^{10}-9n+4}}\cdot\frac{n^2}{1}\right) = \frac{7}{\sqrt{49}} = 1 > 0$

$\sum b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent by p-Test

$\therefore$ by L.C.T.T. $\Rightarrow \sum a_n$ is also convergent

b) $\sum_{n=1}^{\infty} \frac{7n^{1}-n^{1}+4}{\sqrt{2n^{4}+5n-2}} \overset{\text{let}}{=\!=} a_n.$

$b_n =$ dominant terms of $a_n = \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n} = b_n.$

Consider $C = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{6n+4}{\sqrt{2n^4+5n-2}} \cdot \frac{n}{1} \right) = \frac{6}{\sqrt{2}} > 0$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by P-Test

$\sum_{n=1}^{\infty} \frac{1}{n}$ {Harmonic series.}

$\therefore$ by L.C.T.T. $\Rightarrow \sum a_n$ is divergent.

c) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}; \overset{\text{let}}{=\!=} a_n$

let $b_n =$ dominant terms of $a_n = \frac{1}{2^n}$

Consider $C = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{2^n - 1} \cdot \frac{2^n}{1} \right) = \lim_{n\to\infty} \left(\frac{2^n}{2^n - 1} \div \frac{2^n}{2^n} \right)$

$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n} \to 0} = 1 > 0.$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n \leftarrow$ G.S.; $r = \left|\frac{1}{2}\right| < 1$ is convergent by G.S.

$\therefore$ by L.C.T.T. $\sum a_n$ is convergent.

d) $\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$

Dominant terms: $\frac{\cancel{n}}{\cancel{n}2^n} = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \to$ convergent by G.S. $\Rightarrow$ Construct bigger series.

$$\sum_{n=1}^{\infty} \frac{n+1}{n \cdot 2^n} \le \sum_{n=1}^{\infty} \frac{n+n}{n \cdot 2^n} = \sum \frac{2\cancel{n}}{\cancel{n} \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \quad \begin{cases} a = 1 \\ r = \left|\frac{1}{2}\right| < 1 \end{cases}$$

is convergent by G.S.

$\therefore$ by DCTT $\Rightarrow \sum a_n$ is also convergent.