So far: 1. G.S
2. T.S.
3. D.T.T.
4. I.T.T.
5. P-Test
6. DCTT
7. LCTT.
8. ALT.

### *Alternating Series*

***Def***: An alternating series is a series of the form $a_1 + a_2 - a_3 + a_4 - a_5 + \ldots$
Such as $1 + 2 - 3 + 4 - 5 + 6 \ldots$

(ALT) ***The Alternating Series Test***: Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots \quad (a_n > 0)$

Satisfies the following conditions:

i) $a_{n+1} \le a_n \Rightarrow$ Decreasing without negative sign.

ii) $\lim_{n\to\infty} a_n = 0$

Then the series is convergent.

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \overset{\text{let}}{=\!=} a_n$

$a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ by ALT. $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$. is convergent

b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ this is an alternating series, but ~~$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1}$~~ $= \frac{3}{4} \ne 0$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3n}{4n-1} \overset{\text{let}}{=\!=} a_n.$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \ne 0 \Rightarrow$ ALT fails.

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{4n-1} \overset{\text{let}}{=\!=} a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n \cdot 3n}{4n-1} = \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty}\left(\frac{3n}{4n-1}\right)$

$= \text{DNE} \cdot \frac{3}{4}$ " $=$ DNE $\ne 0$

$(-1)^{1000001} = -1$

... is divergent.

$\therefore$ By D.T.T. $\Rightarrow \sum a_n$ is divergent.

c) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2^n} \stackrel{\text{let}}{=} a_n$.

i) $a_n = \frac{1}{2^n}$ ; $a_{n+1} = \frac{1}{2^{n+1}}$

$a_n = \frac{1}{2^n} \geq \frac{1}{2^{n+1}} = a_{n+1}$.

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$

By A.L.T., $\Rightarrow \sum (-1)^n a_n$ is convergent.

d) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$; Let $a_n = \frac{n^2}{n^3+1}$; How do we know $a_n = \frac{n^2}{n^3+1}$ is decreasing, consider the following function $f(x) = \frac{x^2}{x^3+1} \Rightarrow f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} \Rightarrow f'(x) < 0$ for $x > \sqrt[3]{2}$ i.e.

$a_n = \frac{n^2}{n^3+1}$ is decreasing. ✓

$\lim_{n\to\infty}\frac{n^2}{n^3+1} = 0$. By the Alternating Series Test, $a_n = \frac{n^2}{n^3+1}$ is convergent.

e) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{3n+2}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{3n+2}}$ let $a_n$.

i) $a_n = \frac{1}{\sqrt{3n+2}} \geq a_{n+1} = \frac{1}{\sqrt{3(n+1)+2}}$.

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{3n+2}} = 0$

By A.L.T. $\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$ is convergent

f) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^4+5}$ ✓

## Error Analysis for A.L.T.

***Estimating Sums***:

***Alternating Series Estimation Theorem***: If $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is the sum of an alternating series that satisfies

i) $0 \le a_{n+1} \le a_n$ and ii) $\lim_{n\to\infty} a_n = 0$ Then $|R_n| = |S - S_n| \le |a_{n+1}|$

$$\text{Error} = |R_n| \le |a_{n+1}|$$

***Ex***: Approximate the sum of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ with an error of less than 0.01.

$\to \sum (-1)^{n-1} \frac{1}{n}$

Sol: Error $= |R_n| < 0.01$.

where $|R_n| \le |a_{n+1}|$, where $a_n = \frac{1}{n}$.

$|R_n| \le \left|\frac{1}{n+1}\right| < 0.01$

$\frac{1}{n+1} < 0.01 = \frac{1}{0.01} < n+1$

$n > 100 - 1 = 99$.

Ex: How many terms are needed in computing the sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2n+4}$ to ensure its accuracy to 0.001

Sol: $n = ?$ such that $|\text{Error}| < 0.001$

where $|\text{Error}| = |R_n| < |a_{n+1}|$.

where $a_n = \frac{1}{n^3+2n+4}$.

$|\text{Error}| = |R_n| < \frac{1}{(n+1)^3 + 2(n+1) + 4} < 0.001 = \frac{1}{1000}$

Trial & error:

Pick $n = 5 \Rightarrow \frac{1}{6^3+12+4} = 0.00431$

?

pick $\boxed{n = 49}$ ✓ $\Rightarrow$ $\dfrac{1}{(50)^3 + 2(50) + 4} < 0.001$.

$\underbrace{(50)^3 + 2(50) + 4}$ " $0.0000007$

$\boxed{n = 10}$ $\Rightarrow$ $\dfrac{1}{10^3 + 2(10) + 4} = \dfrac{1}{1024} = 0.0009 < 0.001$

—— // —— Q / —— // ——

Error Analysis. ✓

For I.T.T. $\text{Error} = |R_n| \leq \int_n^\infty f(x)\,dx$

For ALT. $\text{Error} = |R_n| \leq |a_{n+1}|$.

$\sum \dfrac{n}{(5n^2+1)^7} \Rightarrow f(x) = \dfrac{x}{(5x^2+1)^7}$ ← I.T.T.

$\sum \dfrac{(-1)^{n-1}}{(5n^2+1)^7} \Rightarrow a_n = \dfrac{1}{(5n^2+1)^7}$ ← ALT.

Error: $\int_n^{\infty} \frac{x}{(5x^2+1)^7}\,dx \leq 0.001$

Error: $a_{n+1} = \frac{1}{(5(n+1)^2+1)^7} \leq 0.001$

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$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+5n}}$ ← ALT.

$n = ?$ such that error $\leq 0.001$

$|R_n| \leq |a_{n+1}|$ where $a_n = \frac{1}{\sqrt{n^2+5n}}$

$|R_n| \leq \frac{1}{\sqrt{(n+1)^2+5(n+1)}} \leq 0.001$

$10^{-3}$

Trial: error $n = 99$

$\frac{1}{\sqrt{100^2+5(100)}} = 0.0097$

$n = 999$. $\frac{1}{\sqrt{\quad}} = 0.00097 < 0.001$

$\sqrt{1000^2 + 5(1000)}$