Section 7.2 *Trigonometric Integrals*

For $\int \sin^m x \cos^n x\,dx \Rightarrow \begin{cases} \text{if } m \text{ is odd} \Rightarrow \text{let } u = \cos(x) \\ \text{if } n \text{ is odd} \Rightarrow \text{let } u = \sin(x) \end{cases}$

let u be the "other function"
If (cosine)^odd ⟹ u = sine
(sine)^odd ⟹ u = cosine.

Ex: Integrate the following:

a) $\int \sin^4(3x)\cos^5(3x)\,dx = \int \sin^4(3x)\cdot\cos^4(3x)\cdot \underbrace{\boxed{\cos(3x)dx}}_{\frac{du}{3}}$

let $u = \sin(3x)$

$du = 3\cos(3x)dx$

$\frac{du}{3} = \cos(3x)dx$

$= \int \sin^4(3x)\cdot(1-\sin^2(3x))^2\cdot\cos(3x)\,dx.$

$= \int u^4(1-u^2)^2\cdot\frac{du}{3} = \frac{1}{3}\int u^4(1-2u^2+u^4)\,du.$

$= \frac{1}{3}\int(u^4 - 2u^6 + u^8)\,du = \frac{1}{3}\left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9\right] + C$

$= \frac{1}{3}\left[\frac{1}{5}\sin^5(3x) - \frac{2}{7}\sin^7(3x) + \frac{1}{9}\sin^9(3x)\right] + C$

b) $\int \sqrt[5]{\cos^3(2x)}\,\sin^3(2x)\,dx = \int \sqrt[5]{\cos^3(2x)}\cdot\sin^2(2x)\cdot\sin(2x)dx$

let $u = \cos(2x)$

$du = -2\sin(2x)dx$

$\frac{du}{-2} = \sin(2x)dx$

$= \int \sqrt[5]{\cos^3(2x)}\cdot(1-\cos^2(2x))\sin(2x)dx$

$= \int \sqrt[5]{u^3}\cdot(1-u^2)\frac{du}{-2}$

$= -\frac{1}{2}\int(u^{3/5} - u^{13/5})\,du = -\frac{1}{2}\left[\frac{5}{8}u^{8/5} - \frac{5}{18}u^{18/5}\right] + C.$

$= -\frac{1}{2}\left[\frac{5}{8}(\cos(2x))^{8/5} - \frac{5}{18}(\cos(2x))^{18/5}\right] + C.$

$\int \frac{\cos^5(3x)}{\sqrt[4]{\sin^3(3x)}}\,dx = \int \frac{\cos^4(3x)}{\sqrt[4]{\sin^3(3x)}}\cdot\underbrace{\cos(3x)dx} = \int \frac{(1-\sin^2(3x))^2}{\sqrt[4]{\sin^3(3x)}}\cdot\cos(3x)dx$

Let $u = \sin(3x)$

$du = 3\cos(3x)dx$

$\frac{du}{3} = \boxed{\cos(3x)dx}$

$= \int \frac{(1-u^2)^2}{u^{3/4}}\cdot\frac{du}{3}$

$= \frac{1}{3}\int \frac{1-2u^2+u^4}{u^{3/4}}\,du = \frac{1}{3}\int(u^{-3/4} - 2u^{5/4} + u^{13/4})\,du.$

$= \frac{1}{3}\left[4(\sin(3x))^{1/4} - 2\cdot\frac{4}{9}(\sin(3x))^{9/4} + \frac{4}{17}(\sin(3x))^{17/4}\right] + C$

What if none of the power is odd ➔ $\boxed{\text{Double angle formulas}}$

Remember These.

$$\cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$$

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

$$\boxed{\cos(2\theta) = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta} \Rightarrow \boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \Rightarrow \boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

Double Angle Formulas.

c) $\displaystyle\int \cos^4(3x)\,dx = \int\left(\cos^2(3x)\right)^2 dx = \int\left(\frac{1+\cos(6x)}{2}\right)^2 dx = \frac{1}{4}\int\left[1 + 2\cos(6x) + \cos^2(6x)\right]dx.$

$\frac{1}{2} + \frac{1}{2}\cos(12x)$

$$= \frac{1}{4}\int\left[1 + 2\cos(6x) + \left(\frac{1+\cos(12x)}{2}\right)\right]dx.$$

$$= \frac{1}{4}\int\left[\frac{3}{2} + 2\cos(6x) + \frac{1}{2}\cos(12x)\right]dx.$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{3}\sin(6x) + \frac{1}{24}\sin(12x)\right] + C$$

For tangent / cotangent / secant / cosecant functions

Ex: Evaluate

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C.$$

$\tan^2\theta = \sec^2\theta - 1$

Ex: Integrate the following:

a) $\int \sec^3(4x)\tan^3(4x)\,dx = \int \sec^2(4x)\cdot\tan^2(4x)\cdot\sec(4x)\tan(4x)\,dx$

let $u = \sec(4x)$

$du = 4\sec(4x)\cdot\tan(4x)\cdot dx$

$\frac{du}{4} = \sec(4x)\tan(4x)\,dx$

$= \int \sec^2(4x)\left(\sec^2(4x) - 1\right)\sec(4x)\tan(4x)\,dx$

$= \int u^2(u^2-1)\frac{du}{4} = \frac{1}{4}\int (u^4 - u^2)\,du$

$= \frac{1}{4}\left[\frac{1}{5}\sec^5(4x) - \frac{1}{3}\sec^3(4x)\right] + C$

b) $\int \sec^4(3x)\tan^4(3x)\,dx = \int \sec^2(3x)\cdot\tan^4(3x)\cdot\sec^2(3x)\,dx$

let $u = \tan(3x)$

$du = 3\sec^2(3x)\,dx$

$\frac{du}{3} = \sec^2(3x)\,dx$

$= \int \left(\tan^2(3x)+1\right)\cdot\tan^4(3x)\cdot\sec^2(3x)\,dx$

$= \int (u^2+1)\cdot u^4\cdot\frac{du}{3}$

$= \frac{1}{3}\int (u^6 + u^4)\,du$

$= \frac{1}{3}\left[\frac{1}{7}\tan^7(3x) + \frac{1}{5}\tan^5(3x)\right] + C$

c) $\int \sec^4(2x)\tan^3(2x)\,dx = \int \sec^2(2x)\tan^3(2x)\cdot\sec^2(2x)\,dx$

let $u = \tan(2x)$

$du = 2\sec^2(2x)\,dx$

$\frac{du}{2} = \sec^2(2x)\,dx$

$= \int \left(\tan^2(2x)+1\right)\cdot\tan^3(2x)\cdot\sec^2(2x)\,dx$

$= \int (u^2+1)\cdot u^3\cdot\frac{du}{2}$

$= \frac{1}{2}\int (u^5 + u^3)\,du = \frac{1}{2}\left[\frac{1}{6}\tan^6(2x) + \frac{1}{4}\tan^4(2x)\right] + C$

d) $\int \tan^3(4x)\,dx$ $= \int \tan^2(4x) \cdot \tan(4x) \cdot dx$

or: $\int \frac{\sin^3(4x)}{\cos^3(4x)}dx$

let $u = \cos(4x)$

$$= \int \left(\sec^2(4x) - 1\right) \cdot \tan(4x)\,dx$$

$$= \underbrace{\int \sec^2(4x) \cdot \tan(4x)\,dx}_{} - \underbrace{\int \tan(4x)\,dx}_{}$$

$u = \tan(4x)$

$\frac{du}{4} = \sec^2(4x)\,dx$

$$= \frac{1}{4}\int u\,du - \frac{1}{4}\ln|\sec(4x)| + C.$$

$$= \frac{1}{8}\tan^2(4x) - \frac{1}{4}\ln|\sec(4x)| + C.$$

d) $\int \csc^4(5x)\,dx$

$$= \int \underbrace{\csc^2(5x)}_{} \cdot \csc^2(5x)\,dx$$

$$= \int \left(1 + \cot^2(5x)\right) \cdot \csc^2(5x)\,dx = \int (1+u^2)\frac{du}{-5}$$

let $u = \cot(5x)$

$du = -5\csc^2(5x)\,dx$

$\frac{du}{-5} = \csc^2(5x)\,dx$

$$= -\frac{1}{5}\int (1+u^2)\,du = -\frac{1}{5}\left[u + \frac{1}{3}u^3\right] + C$$

$$= -\frac{1}{5}\left[\cot(5x) + \frac{1}{3}\cot^3(5x)\right] + C.$$

b) $\int \sec^3 x\,dx$

$\rightarrow$ Reduction formula:

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx.$$

$$\int \sec^3 x\,dx = \underbrace{\frac{\sec x \tan x}{2}}_{\downarrow} + \frac{1}{2}\underbrace{\int \sec x\,dx}_{}.$$

$$= A + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Evaluating $\int \sin(mx)\cos(nx)dx; \int \sin mx \sin nx dx; \int \cos mx \cos nx dx$

a) $\sin A \cos B = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$

b) $\sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$

c) $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

Ex: Evaluate:

a) $\int \sin(7x)\cos(5x)dx = \frac{1}{2}\int\left[\sin(7x-5x)+\sin(7x+5x)\right]dx$

$= \frac{1}{2}\int\left[\sin(2x)+\sin(12x)\right]dx$

$= \frac{1}{2}\left[-\frac{1}{2}\cos(2x)-\frac{1}{12}\cos(12x)\right]+C.$

b) $\int \cos(4x)\cos(3x)dx = \frac{1}{2}\int\left[\cos(4x-3x)+\cos(4x+3x)\right]dx$

$= \frac{1}{2}\int\left[\cos x+\cos(7x)\right]dx.$

$= \frac{1}{2}\left[\sin x+\frac{1}{7}\sin(7x)\right]+C.$