Section 11.6 Absolute convergence and The Ratio and Root Tests

Def: A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

i.e. If $\sum |a_n|$ is convergent $\Rightarrow$ $\sum a_n$ is abs. convergent.

Test for Abs. Convergent.

Ex: ~~The series:~~

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} \xlongequal{\text{let}} a_n.$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3^n} \right| = \sum_{n=1}^{\infty} \left| -\frac{1}{3} \right|^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \quad \left\{ \begin{array}{l} r = \left|\frac{1}{3}\right| < 1 \\ \text{Convergent by} \\ \text{G.S.} \end{array} \right.$$

$\Rightarrow \sum |a_n|$ is convergent $\xRightarrow[\text{by def.}]{}$ $\sum a_n$ is abs. convergent

b) $\sum_{n=1}^{\infty} \frac{\sin(5n-\pi)}{n^3} \xlongequal{\text{let}} a_n$ b/c $|\sin(5n-\pi)| \leq 1$

$$\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{\sin(5n-\pi)}{n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \left\{ \begin{array}{l} p = 3 > 1 \\ \text{is convergent} \\ \text{by } p\text{-Test} \end{array} \right.$$

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum |a_n|$ is convergent.

by def. $\sum_{n=1}^{\infty} a_n$ is abs. convergent.

$\sum a_n$ is convergent / $\sum |a_n|$ is divergent

Def: A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \overset{\text{let}}{=} a_n$

$\sum a_n$ is convergent Alt. Harmonic Series is convergent.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \left\{ \begin{array}{l} p=1 \text{ is} \\ \text{divergent by} \\ p\text{-Test} \end{array} \right.$$

$\Rightarrow$ By def. $\sum a_n$ is conditionally convergent.

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{\sqrt{3n^7 + 2n^5 + 1}} \overset{\text{let}}{=} a_n$.

$$\left\{ \begin{array}{l} 1.\ a_{n+1} \le a_n \\ 2.\ \lim_{n\to\infty} a_n = 0 \end{array} \right.$$

$\sum a_n$ is convergent by A.L.T.

$$\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} \cdot n^3}{\sqrt{3n^7 + 2n^5 + 1}} \right| = \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{3n^7 + 2n^5 + 1}} \ge \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{3n^7 + 2n^7 + n^7}}$$

$$= \sum \frac{n^3}{\sqrt{6n^7}} = \frac{1}{\sqrt{6}} \sum \frac{1}{n^{\frac{7}{2}-3}} = \frac{1}{\sqrt{6}} \sum \frac{1}{n^{1/2}} \quad p = \tfrac{1}{2} < 1$$

divergent by P-Test

$\therefore$ by D.C.T.T. $\Rightarrow \sum |a_n|$ is divergent

By ~~*~~ def. $\sum a_n$ is conditionally convergent.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Ex: Test for convergence / divergence:

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{n^5+2n+7} \overset{\text{let}}{=\!=} a_n$.

$$\sum |a_n| = \sum \left| \frac{(-1)^{n-1}(n+2)}{n^5+2n+7} \right| = \sum \frac{n+2}{n^5+\cancel{2n+7}} \leq \sum_{n=1}^{\infty} \frac{n+2\cdot n}{n^5}$$

$$= \sum \frac{3n}{n^5} = 3\sum \frac{1}{n^4} \left\{ \begin{array}{l} p=4>1 \text{ is} \\ \text{convergent by} \\ p-\text{Test} \end{array} \right.$$

$\therefore$ by D.C.T.T. $\Rightarrow \sum |a_n|$ is Abs. Convergent.

$\Rightarrow \sum a_n$ is convergent.

b) $\sum_{n=1}^{\infty} (-1)^n \left[\ln(2n+1) - \ln(n+2)\right] = \sum_{n=1}^{\infty} (-1)^n \cdot \ln\left(\frac{2n+1}{n+2}\right) \overset{\text{let}}{=\!=} a_n$.

$$\lim_{n\to\infty} a_n = \underbrace{\left[\lim_{n\to\infty} (-1)^n\right]}_{= \text{DNE}} \cdot \underbrace{\left[\lim_{n\to\infty} \ln\left(\frac{2n+1}{n+2}\right)\right]}_{\ln 2} = \text{DNE} \neq 0$$

$\therefore$ by D.T.T $\Rightarrow \sum a_n$ is divergent.

c) $\sum_{n=1}^{\infty} \frac{(-3)^n}{7^n+4^n} \overset{\text{let}}{=\!=} a_n$.

Consider $$\sum |a_n| = \sum \left| \frac{(-3)^n}{7^n+4^n} \right| = \sum_{n=1}^{\infty} \frac{3^n}{7^n+\cancel{4^n}} \leq \sum_{n=1}^{\infty} \frac{3^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n \left\{ \begin{array}{l} r=\left|\frac{3}{7}\right|<1 \\ \text{is convergent} \\ \text{by G.S.} \end{array} \right.$$

$\Rightarrow$ By D.C.T.T. $\Rightarrow \sum |a_n|$ is convergent

$\Rightarrow \sum a_n$ is abs. convergent $\Rightarrow \sum a_n$ is convergent.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

The Ratio Test: Given a series $\sum_{n=1}^{\infty} a_n$ Define $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1$, (Test fails)

Work well with factorials

Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$

Ex: Test for the convergence:

a) $\sum_{n=1}^{\infty} \frac{n}{3^n} \overset{\text{let}}{=} a_n$. $a_n = \frac{n}{3^n} \Rightarrow a_{n+1} = \frac{n+1}{3^{n+1}}$

Consider $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{3n} \right| = \frac{1}{3} < 1$

By Ratio-Test $\Rightarrow \sum a_n$ is convergent.

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{3^n} \overset{\text{let}}{=} a_n$.

Consider $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n}^{(-1)} \cdot (n+1)^3}{\cancel{3^{n+1}}_{3}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^{n-1}} \cdot n^3} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{3n^3} \right| = \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = \frac{1}{3} < 1$

By Ratio-Test $\Rightarrow \sum a_n$ is convergent.

c) $\sum_{n=1}^{\infty} \frac{n^n}{n!} \overset{\text{let}}{=} a_n$. $\quad \underbrace{a_{n+1}}_{} \quad \underbrace{\frac{1}{a_n}}_{}$

$$\text{Consider } L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{\cancel{n+1}} \cdot \cancel{n!}}{(n+1)^{\cancel{1}} \cdot \cancel{n!} \cdot n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}.$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

By Ratio-Test $\Rightarrow \sum a_n$ is divergent

d) $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \leq \sum \frac{1 \cdot 2^n + 5 \cdot 2^n}{3^n} = \sum \frac{6 \cdot 2^n}{3^n}$

$$= \sum 6 \left(\frac{2}{3}\right)^n \quad \left\{ \begin{array}{l} r = \left|\frac{2}{3}\right| < 1 \\ \text{is convergent} \\ \text{by G.S.} \end{array} \right.$$

$\therefore$ by D.C.T.T. $\Rightarrow \sum \frac{2^n + 5}{3^n}$ is convergent

$$a_n = \frac{(2n)!}{n!\cdot n!} = \frac{(2n)!}{(n!)^2}$$

f) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \stackrel{\text{let}}{=} a_n.$

Consider $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(2(n+1))!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!}\right]$

$$5! = 5\cdot 4!$$

$$= \lim_{n\to\infty} \frac{(2n+2)!\cdot(n!)^2}{[(n+1)\cdot n!]^2\cdot(2n)!}$$

Note: $(2n+2)! = (2n+2)(2n+1)(2n)!$

$$= \lim_{n\to\infty}\frac{\underbrace{(2n+2)}_{2(n+1)}(2n+1)\cdot\cancel{(2n)!}\cdot\cancel{(n!)^2}}{(n+1)^{\cancel{2}}\,\cancel{(n!)^2}\cdot\cancel{(2n)!}} = \lim_{n\to\infty}\frac{2(2n+1)}{n+1} = \lim_{n\to\infty}\frac{4n+4}{n+1} = 4 > 1$$

By the Ratio-Test $\sum a_n$ is divergent.

g) $\displaystyle\left(\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}\right)$

h) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{(2\cdot 4\cdot 6\cdots(2n))\left[4^{n+2}+3\right]} = a_n.$

$n=1$ $\quad$ $n=1$ $\quad$ Extra.

Consider $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$

$$\lim_{n\to\infty}\left|\underbrace{\frac{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}\,(2(n+1)+1)}{\cancel{2\cdot 4\cdot 6\cdots(2n)}\,(2(n+1))\left[4^{n+1+2}+3\right]}}_{=a_{n+1}}\cdot\underbrace{\frac{\cancel{2\cdot 4\cdot 6\cdots(2n)}\left[4^{n+2}+3\right]}{\cancel{1\cdot 3\cdot 5\cdots(2n+1)}}}_{=\frac{1}{a_n}}\right|$$

$n=1$, $n=2$, $n=3$, $n=n$, $n=n+1$

$$\Rightarrow L = \lim_{n\to\infty}\frac{(2n+3)\left(4^{n+2}+3\right)}{(2n+2)\left[4^{n+3}+3\right]} = \underbrace{\lim_{n\to\infty}\left(\frac{2n+3}{2n+2}\right)}_{=1}\cdot\lim_{n\to\infty}\frac{4^{n+2}+3}{4^{n+3}+3}\cdot\frac{\frac{1}{4^{n+3}}}{\frac{1}{4^{n+3}}}$$

$$= 1\cdot\lim_{n\to\infty}\frac{\frac{1}{4}+\cancel{\frac{3}{4^{n+3}}}^{\,0}}{1+\cancel{\frac{3}{4^{n+3}}}_{\,0}} = 1\cdot\frac{1}{4} = \frac{1}{4}<1$$

By Ratio - Test $\Rightarrow \sum a_n$ is convergent.

The Root Test: Given a series $\sum_{n=1}^{\infty}a_n$ $\to$ Let $L=\lim_{n\to\infty}\sqrt[n]{|a_n|}$ $\quad$ If $\boxed{a_n=[f(n)]^n}$

i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L<1$, then the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent (hence it's convergence)

ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L>1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\infty$, then the series $\sum_{n=1}^{\infty}a_n$ is divergent.

iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=1$, the Root Test is inconclusive. (Root test fails).

Ex: Test the convergence /divergence of the following series:

a) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^{n}$ $\overset{\text{let}}{=\!=}$ $a_n = (f(n))^n$.

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left|\frac{2n+3}{3n+2}\right|^{\cancel{n}}} = \lim_{n\to\infty} \left|\frac{2n+3}{3n+2}\right| = \frac{2}{3} < 1$$

$\therefore$ By Root-Test $\Rightarrow \sum a_n$ is convergent.

b) $\sum_{n=1}^{\infty}\left(\frac{7n^2+5n+4}{3n^2+2n+4}\right)^{n}$ $\overset{\text{let}}{=\!=}$ $a_n = (f(n))^n$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left|\frac{7n^2+5n+4}{3n^2+2n+4}\right|^{\cancel{n}}} = \lim_{n\to\infty} \left|\frac{7n^2+5n+4}{3n^2+2n+4}\right| = \frac{7}{3} > 1$$

By Root-Test $\Rightarrow \sum a_n$ is divergent.

b) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{2n+5}\right)^{n^2}$ $\overset{\text{let}}{=\!=}$ $a_n = (f(n))^{n^2} = \left[(f(n))^n\right]^n$.

Consider $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left[\left(\frac{2n+3}{2n+5}\right)^n\right]^{\cancel{n}}}$

$$= \lim_{n\to\infty} \left(\frac{2n+3}{2n+5}\div\frac{2n}{2n}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{3}{2n}\right)^n}{\left(1+\frac{5}{2n}\right)^n} = \frac{e^{3/2}}{e^{5/2}}$$

$$= \frac{1}{e^{\frac{5}{2}-\frac{3}{2}}} = \frac{1}{e} < 1$$

Short-Cut

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$\therefore$ by Root - Test $\Rightarrow \sum a_n$ is convergent.

For Series.

1. G.S.
2. T.S.
3. D.T.T.
4. I.T.T. $\longleftarrow$ Error $|R_n| \le \int_n^\infty f(x)\,dx$
5. P- Test.
6. D.C.T.T.
7. L.C.T.T.
8. A.L.T. $\longleftarrow$ Error $|R_n| \le |a_{n+1}|$.
9. Abs. Convergent.
10. Ratio - Test
11. Root - Test
