## Section 11.8 Power Series

***Def***: A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \dots$

$\sum_{n=0}^{\infty} c_n (x-0)^n = \sum_{n=0} c_n x^n$ : power series centered at $x=0$

$x=0$ : center.

General form: $\sum_{n=0}^{\infty} c_n (x-a)^n$ : power series centered at a or a power series about a.

$x=a$

We know that $\sum_{n=0}^{\infty} x^n$ is convergence if and only if $|x|<1$ $\rightarrow$ $\underbrace{-1<x<1}_{\text{Interval of Convergence}}$.

G.S.

**Ex**: Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^n \stackrel{\text{let}}{=} a_n$.

To find the interval of convergence $\Rightarrow$ Use Ratio-Test.

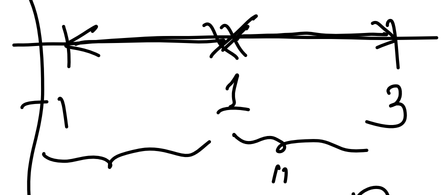
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x-1)^n} \right|$$

$x-1$ $2$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cdot(x-1)}{2n} \right| = \left| \frac{x-1}{2} \right| \cdot \underbrace{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)}_{=1}$$

$$L = \left| \frac{x-1}{2} \right| < 1 \Rightarrow -1 < \frac{x-1}{2} < 1 \Rightarrow -2 < x-1 < 2$$

$$\boxed{-1 < x < 3}$$




2 : Radius of convergence.

***Ex***: Find interval of convergence: for convergence

a) $\sum_{n=0}^{\infty} n!\,x^n \stackrel{\text{let}}{=} a_n$,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\cdot x^n}\right|.$$

$(n+1)(n!)$ $\quad x$

$$= \lim_{n\to\infty}\left|(n+1)\,x\right|$$

for $\forall x \neq 0$ $\qquad$ ~~R~~ $\quad L<1$

$$= |x| \lim_{n\to\infty}(n+1) = \infty$$

Interval of convergence $\Rightarrow$ $|x| = 0 < 1$
$\{x=0\}$

b) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \stackrel{\text{let}}{=} a_n$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|.$$

$(n+1)! = (n+1)\cdot n!$ $\quad x$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1 \text{ for all } x.$$

Interval of convergence : $(-\infty, \infty)$.

***Theorem***: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

i) The series converges only when x = a.
ii) The series converges for all x. $(-\infty,\infty)$
iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

R : Radius of convergence. $\quad$ a-R $\quad$ a $\quad$ a+R. $\quad$ R $\quad$ R

Ex: Determine interval of convergence:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^n$

b) $\displaystyle\sum_{n=0}^{\infty} 2^n \cos^n x$

c) $J_0 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \underline{\underline{\text{let}}}\ a_n.$ ✓ $x^{2n+2} = x^{2n} \cdot x^2$

$(-1)$ $x^2$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} \cdot x^{2(n+1)}}{2^{2(n+1)} \cdot ((n+1)!)^2} \cdot \frac{\cancel{2^{2n}} \cdot \cancel{(n!)^2}}{\cancel{(-1)^n} \cdot \cancel{x^{2n}}} \right|.$$

$2^{2(n+1)} = \cancel{2^{2n}} \cdot 2^2 \cdot (n+1)^2 \cancel{(n!)^2}$

$$\Rightarrow L = \lim_{n\to\infty} \left| \frac{x^2}{4(n+1)^2} \right| = \left| \frac{x^2}{4} \right| \cdot \underbrace{\lim_{n\to\infty} \frac{1}{(n+1)^2}}_{=0} = 0 < 1 \text{ for all } x$$

Interval of convergence: $(-\infty, \infty)$

d) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} \underline{\underline{\text{let}}}\ a_n$

$=(-3)$ $=x$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} \cdot x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{(-3) \cdot x \cdot \sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= |3x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = |3x| \underbrace{\lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}}_{=1} = |3x|$$

for convergence $\Rightarrow L < 1 \Rightarrow |3x| < 1 \Rightarrow -1 < 3x < 1$

$$\boxed{-\frac{1}{3} < x < \frac{1}{3}}$$

e) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} \underline{\underline{\text{let}}}\ a_n.$ center: $x=-2$

$x+2$

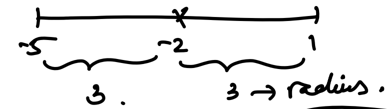
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+2)^{n+1}}{\cancel{3^{n+2}}} \cdot \frac{\cancel{3^{n+1}}}{n\,\cancel{(x+2)^n}} \right|.$$

$3$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(x+2)}{3n} \right| = \left| \frac{x+2}{3} \right| \cdot \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{=1} = \left| \frac{x+2}{3} \right|$$

$$L = \left| \frac{x+2}{3} \right| < 1 \Rightarrow -1 < \frac{x+2}{3} < 1$$

$$\Rightarrow \quad -3 < x+2 < 3 \quad \Rightarrow \quad \boxed{-5 < x < 1}$$




## *Section 12.9 Representations of Functions as Power Series*

$$\sum_{n=1}^{\infty} ax^{n-1} = \frac{a}{1-x} \text{ for } |x| < 1$$

$$\boxed{\sum_{n=0}^{\infty} a\cdot r^n = \frac{a}{1-r} \text{ if } |r|<1} = \sum_{n=0}^{\infty} 1\cdot x^n = \boxed{\sum x^n = \frac{1}{1-x}}$$

let $a = 1$ ; $r = x$

Given $f(x) = \sum a_n x^n$.

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1$$

$$\boxed{f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ if } |x| < 1}$$

General form: $f(x) = \frac{1}{1-A(x)} = \sum_{n=0}^{\infty} A^n$ for $|A| < 1$ where $A$ is any expression in term of $x$.

***Ex***: Express the following expression as the sum of a power series and find the interval of convergence.

a) $f(x) = \dfrac{1}{1+2x^2} = \dfrac{1}{1 \mp A(x)}$

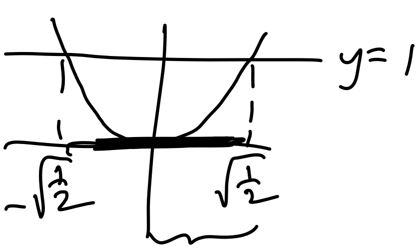
$$f(x) = \frac{1}{1-\underbrace{(-2x^2)}_{A(x)}} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot x^{2n}$$

Interval of Convergence: $|A(x)| < 1$

$$\Rightarrow \quad |-2x^2| < 1$$

$$|2x^2| < 1$$

$y = 2x^2$




$$-1 < 2x^2 < 1$$

$$-\frac{1}{2} < x^2 < \frac{1}{2}$$

$$-\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}$$

$$0 < x^2 < \frac{1}{2}$$

$$0 < x < \sqrt{\frac{1}{2}}$$

$$= \boxed{\frac{1}{1-A} = \sum A^n \quad |A|<1}$$

(b) $f(x) = \dfrac{9x-7}{x^2-x-6}$ about the center $x=8$

$$f(x) = \frac{9x-7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \qquad \begin{cases} A\big|_{x=3} = \frac{27-7}{5} = 4 \\ B\big|_{x=-2} = \frac{-18-7}{-5} = 5 \end{cases}$$

$$f(x) = \frac{4}{x-3} + \frac{5}{x+2}.$$

Taking care of the center $x=8 \Rightarrow (x-8)$

$(x-8)$

$$= \frac{4}{(x-8)+5} + \frac{5}{(x-8)+10}$$

$$\frac{1}{1-A} = \sum A^n$$

$$= \frac{4}{5} \cdot \frac{1}{1-\left(-\frac{x-8}{5}\right)} + \frac{5}{10} \cdot \frac{1}{1-\left(-\frac{x-8}{10}\right)}$$

$$\frac{4}{5}\sum_{n=0}^{\infty}\left(-\frac{x-8}{5}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-8}{10}\right)^n = \frac{4}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}(x-8)^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{10^n}(x-8)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{4}{5^{n+1}} + \frac{1}{2}\cdot\frac{1}{10^n}\right](x-8)^n$$

***Theorem***: If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, then the function f defined by $\Rightarrow f(x)$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ ; $f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$

ii) $\int f(x)dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

$$\int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx$$

$$= \sum c_n \frac{(x-a)^{n+1}}{\ldots} + C$$

$$\sum_{n=0} \quad \frac{\quad}{n+1}$$

***Ex:*** Find a power series of representation and interval of convergence

a) $f(x) = \ln(1+4x^7)$

$$f(x) = \ln(1+4x^7) = \int \frac{28x^6}{1+4x^7} \cdot dx = 28 \int x^6 \cdot \frac{1}{1-(-4x^7)} dx$$

$$= 28 \int x^6 \sum_{n=0}^{\infty} (-4x^7)^n dx = 28 \int x^6 \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot x^{7n} dx$$

$$= 28 \int \sum_{n=0}^{\infty} (-1)^n \cdot 4^n x^{7n+6} dx = 28 \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \int x^{7n+6} dx$$

$$= \cancel{7} \cdot 4 \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot \frac{x^{7n+7}}{7n+7} + C = \boxed{\sum_{n=0}^{\infty} (-1)^n 4^{n+1} \cdot \frac{x^{7n+7}}{n+1} + C}$$

$$7n+7 = \cancel{7}(n+1)$$

Need to find $C = ?$

$$f(0) = \ln 1 = 0 = 0 + C \Rightarrow C = 0$$

$$\boxed{f(x) = \ln(1+4x^7) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^{n+1}}{n+1} x^{7n+7}}$$

b) ~~$f(x) = \dfrac{1}{(1-5x^3)^2}$~~

Interval of convergence! $|A(x)| < 1$

$$|-4x^7| < 1$$

$$-1 < 4x^7 < 1$$

$$-\frac{1}{4} < x^7 < \frac{1}{4}$$

$$\boxed{-\sqrt[7]{\frac{1}{4}} < x < \sqrt[7]{\frac{1}{4}}}$$

c) $f(x)=\tan^{-1}x = \int \frac{1}{1+x^2}dx = \int \frac{1}{1-(-x^2)}dx$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n dx. = \int \sum_{n=0}^{\infty} (-1)^n . x^{2n} dx.$$

$$= \sum_{n=0}^{\infty} (-1)^n . \int x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+1}}{2n+1} + C$$

$$f(x) = \tan^{-1}(x) = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n+1}}{2n+1}} + C$$

$$f(0) = \tan^{-1}(0) = 0 = 0 + C \Rightarrow C = 0.$$

$$f(x) = \tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

IOC: $|A(x)|<1$

$|-x^2|<1$

$-1 < x^2 < 1$

$\boxed{-1 < x < 1}$

***Ex***: Evaluate the following as a power series:

a) $\int \frac{x^{1.4}}{2+x^7}dx = \int \frac{x^{1.4}}{2} \cdot \frac{1}{1-\left(-\frac{x^7}{2}\right)} dx$

$$= \frac{1}{2}\int x^{1.4} . \sum_{n=0}^{\infty} \left(-\frac{x^7}{2}\right)^n dx$$

$$= \frac{1}{2}\int x^{1.4} \sum_{n=0}^{\infty} \frac{(-1)^n . x^{7n}}{2^n} . dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \int x^{7n+1.4} dx.$$

$$\boxed{\int \frac{x^{1.4}}{2+x^7} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} . \frac{x^{7n+2.4}}{7n+2.4} + C} \cdot \text{Ans}.$$

b) $\int \tan^{-1}(7x^{13})\, dx$

Know: $\tan^{-1}x = \sum \frac{(-1)^n \cdot x^{2n+1}}{2n+1}$ | $\tan^{-1}(B(x)) = \sum \frac{(-1)^n (B(x))^{2n+1}}{2n+1}$

$$\rightarrow \int \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (7x^{13})^{2n+1}}{2n+1}\, dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 7^{2n+1}}{2n+1} \int x^{26n+13}\, dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 7^{2n+1}}{2n+1} \cdot \frac{x^{26n+14}}{26n+14} + C.$$

c) $\int \boxed{\ln(1+4x^3)}\, dx$

$$= \int \boxed{\int \frac{12x^2}{1+4x^3}\, dx}\, dx$$

$$= \iint 12x^2 \cdot \frac{1}{1-(-4x^3)}\, dx\, dx$$

$$= 12 \iint x^2 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 4^n x^{3n}\, dx\, dx$$

$$= \underbrace{12}_{3\cdot 4} \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \int \left[ \int x^{3n+2}\, dx \right] dx$$

$$= 3 \sum (-1)^n \cdot 4^{n+1} \int \left[ \frac{x^{3n+3}}{3(n+1)} + C \right] dx.$$

$$= 3 \sum_{n=0}^{\infty} \left[ (-1)^n \cdot \frac{4^{n+1}}{3(n+1)} \cdot \frac{x^{3n+4}}{3n+4} + Cx + D. \right]$$