Section 11.10 Taylor and Maclaurin Series

We have the power series as $\sum_{n=0}^{\infty} c_n (x-a)^n$. If we define a function $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$

Given $\boxed{\text{any}}$ function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$; $\boxed{c_n = ? \quad n = 0 \dots \infty}$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$

$f(a) = c_0 \Rightarrow c_0 = \frac{f(a)}{0!}$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$

$f'(a) = c_1 \Rightarrow c_1 = f'(a) = \frac{f'(a)}{1!}$

$f''(x) = 2c_2 + 2\cdot 3 c_3 (x-a) + 3\cdot 4 c_4 (x-a)^2 + 4\cdot 5 c_5 (x-a)^3 + \dots$

$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$

$f'''(x) = 2\cdot 3 c_3 + 2\cdot 3\cdot 4 c_4 (x-a) + 3\cdot 4\cdot 5 c_5 (x-a)^2 + 4\cdot 5\cdot 6 c_6 (x-a)^3 + \dots$

$f'''(a) = 2\cdot 3\cdot c_3 \Rightarrow c_3 = \frac{f'''(a)}{1\cdot 2\cdot 3} = \frac{f'''(a)}{3!}$

Def: Let $f(x)$ has a power representation (expansion) at $x=a$. Where

$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ *where* $c_n = \frac{f^{(n)}(a)}{n!}$ is called a Taylor expansion of f(x) at $x=a$

➔ $\boxed{f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$

Note: The Taylor polynomials of degree n at center $x=a$ $T_n(x,c)=\sum_{i=0}^{n} c_i (x-a)^i$

Def: Taylor expansion of f(x) at $x=0$ is called Maclaurin Series

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex: Find the Taylor series of the following function at the center x = a.

a) $f(x)=\frac{1}{x}$ at $a=2$.

$$f(x)=\frac{1}{x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n \quad = ?$$

$n=0 \Rightarrow f(x)=\frac{1}{x} \Rightarrow f(2)=\frac{1}{2} \quad + \quad = 0!$

$n=1 \Rightarrow f'(x)=-x^{-2} \Rightarrow f'(2)=-2^{-2}=-\frac{1}{2^2} \quad = 1!$

$n=2 \Rightarrow f''(x)=2x^{-3} \Rightarrow f''(2)=2\cdot 2^{-3}=\frac{2}{2^3} \quad 2!$

$n=3 \Rightarrow f'''(x)=-2\cdot 3\cdot x^{-4} \Rightarrow f'''(2)=\frac{-2\cdot 3}{2^4} \quad 3!$

$n=4 \Rightarrow f^{(4)}=2\cdot 3\cdot 4x^{-5} \Rightarrow f^{(4)}(2)=\frac{1\cdot 2\cdot 3\cdot 4}{2^5}=\frac{4!}{2^5}$

$\vdots$

$n=n \longrightarrow f^{(n)}(2)=\frac{(-1)^n n!}{2^{n+1}}$

$$=f^{(n)}(2)$$

$$f(x)=\frac{1}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot \cancel{n!}}{2^{n+1}\cdot\cancel{n!}}(x-2)^n$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

b) $f(x)=\sin(x)$ at $a=\frac{\pi}{3}$?

$$f(x)=\sin x=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(\frac{\pi}{3})}{n!}\left(x-\frac{\pi}{3}\right)^{n}.$$

$n=0 \Rightarrow f(\frac{\pi}{3})=\sin(\frac{\pi}{3})=\frac{\sqrt{3}}{2}$

$n=1 \Rightarrow f'(\frac{\pi}{3})=\cos(\frac{\pi}{3})=\frac{1}{2}$

$n=2 \Rightarrow f''(\frac{\pi}{3})=-\sin(\frac{\pi}{3})=-\frac{\sqrt{3}}{2}$

$n=3 \Rightarrow f'''(\frac{\pi}{3})=-\cos(\frac{\pi}{3})=-\frac{1}{2}$

$n=4 \Rightarrow f^{(4)}(\frac{\pi}{3})=\sin(\frac{\pi}{3})=\frac{\sqrt{3}}{2}$

$\frac{1}{2}$

$n=5$ $\quad n=0 \quad -\frac{1}{2}$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\frac{\sqrt{3}}{2}}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n}$$

$$+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\frac{1}{2}}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

?

$$f(x) = \sum \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex: Find $\boxed{\text{Maclaurin}}$ series of the following functions:

a) $f(x) = e^x$

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$f^{(n)}(0) = ?$

$n=0 \Rightarrow f(x) = e^x \Rightarrow f(0) = e^0 = 1.$

$n=1 \Rightarrow f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$n=2 \Rightarrow f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$\vdots$

$n=n \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$

Know this.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \checkmark$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin x = \sum \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

b) $f(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$

$f^{(n)}(0) = ?$

$n=0 \Rightarrow f(0) = \cos(0) = 1$

$n=1 \Rightarrow f'(0) = -\sin(0) = 0$

$n=2 \Rightarrow f''(0) = -\cos(0) = -1$

$n=3 \Rightarrow f'''(0) = \sin(0) = 0$

$n=4 \Rightarrow f^{(4)}(0) = \cos(0) = 1$

12:30

1:30

4:20

c) $f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$f^{(n)}(0) = ?$

$n=0 \Rightarrow f(0) = \sin(0) = 0$

$n=1 \Rightarrow f'(0) = \cos(0) = 1$

$n=2 \Rightarrow f''(0) = -\sin(0) = 0$

$n=3 \Rightarrow f'''(0) = -\cos(0) = -1$

$n=4 \Rightarrow f^{(4)}(0) = \sin(0) = 0$

$n=5$ $\quad 1$

-1

Normally, we only interest at Taylor series up to certain degree n. So we $f(x) = T_n(x) + R_n(x)$, where $R_n(x)$ is the remainder (error) $R_n(x) = |f(x) - T_n(x)|$

Taylor's Theorem: If f is differentiable through order n + 1 in an open interval I containing a, then for each x in I, there exists a number c between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Taylor's Inequality: If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$

Note: $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$ for any real number x.

Ex: Find the Maclaurin series of the function $f(x) = \sin x$. Show that this series converges to $\sin x$ for all real x.

4. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

2. $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$

5. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$

3. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ✓

Ex: Find the Maclaurin series of the following functions:

a) $f(x) = x^3 \cos\left(7x^2\right)$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (7x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 7^{2n} \cdot x^{4n+3}.$$

b) $f(x) = \frac{x^4}{e^{5x^3}} = x^4 e^{-5x^3}$ where $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= x^4 \sum_{n=0}^{\infty} \frac{(-5x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n x^{3n+4}}{n!}$$

$$\sin(x) = \sum \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

c) $f(x) = \dfrac{\sin(3x^3)}{3x^2} = \dfrac{1}{3x^2} \cdot \sin(3x^3)$

$$= \frac{1}{3x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (3x^3)^{2n+1}$$

$$= \frac{1}{3x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot 3^{2n+1} \cdot x^{6n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot 3^{2n} \cdot x^{6n+1}$$

d) $f(x) = x^5 \tan^{-1}(2x^3)$ (We know $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1}$),

$$f(x) = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot (2x^3)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot 2^{2n+1} \cdot x^{6n+3+5} \quad (= 6n+8)$$

1. e^x $\sum \frac{1}{n!}$
2. $\tan^{-1} x$ $\sum \frac{1}{2n+1}$
3. $\cos x$ $\sum \frac{1}{(2n)!}$
4. $\sin e = \frac{1}{(2n+1)!}$

Ex: Using ~~Maclaurin series~~ to evaluate the following: $= 1$

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\pi^{2n+1}}{2^{2n+1}}$ ↳ odd → sine. $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin(x)}$ = ?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n+1} = \sin\left(\frac{\pi}{2}\right) = 1.$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\pi^{2n+1}}{9^n}$ = cosine = $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos(x) = ?$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi^{2n} \cdot \pi}{(2n)!\,(3^2)^n} = \boxed{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n} = \pi \underbrace{\cos\left(\frac{\pi}{3}\right)}_{=\frac{1}{2}} = \boxed{\frac{\pi}{2}}$$

$\to 3^{2n}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^n}{3^{n-1}}$ = Exponential funct. = $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{n! \cdot 3^n \, 3^{-1}} = 3 \sum_{n=0}^{\infty} \frac{\left(-\frac{2}{3}\right)^n}{n!} = 3 \cdot e^{-2/3} = \boxed{\frac{3}{\sqrt[3]{e^2}}}$$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{n/2}}{(2n+1)} = \tan^{-1} x = \sum \frac{(-1)^n}{2n+1} x^{2n+1}$

$$3^{\frac{n}{2}} \overset{??}{=} x^{2n+1}$$

$$3^{\frac{n}{2}} = \left(3^{\frac{1}{2}}\right)^n = (\sqrt{3})^n = (\sqrt{3})^{\frac{2n}{2}} = \left[(\sqrt{3})^{\frac{1}{2}}\right]^{2n}$$

$$= \left(\sqrt[4]{3}\right)^{2n+1-1} = \left(\sqrt[4]{3}\right)^{2n+1} \cdot \left(\sqrt[4]{3}\right)^{-1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt[4]{3}\right)^{2n+1} \boxed{\left(\sqrt[4]{3}\right)^{-1}}}{2n+1} = \frac{1}{\sqrt[4]{3}} \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\sqrt[4]{3}\right)^{2n+1}}$$

$$= \boxed{\frac{1}{\sqrt[4]{3}} \cdot \tan^{-1}\left(\sqrt[4]{3}\right)} = \#.$$

Ex: a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\cdot dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\int x^{2n}dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\cdot\frac{x^{2n+1}}{2n+1}} + C.$$

$= \# ?$

Error

b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

$$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\frac{x^{2n+1}}{(2n+1)}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

ALT.

$$\text{Error} = |R_n| < |a_{n+1}| = \left|\frac{(-1)^{n+1}}{(n+1)!\,(2(n+1)+1)}\right| < 0.001$$

$$\text{Error} : \leq \frac{1}{(n+1)!\,(2n+3)} < 0.001$$

Trial & error: $n = 5 \Rightarrow \dfrac{1}{\underbrace{6!}_{720}\,(13)} = 0.0001$

$$n = 4 \Rightarrow \frac{1}{5!\,(11)} = \frac{1}{120(11)} = 0.0007 < 0.001$$

$$\int_0^1 e^{-x^2}dx \approx \sum_{n=0}^{4} \frac{(-1)^n}{n!\,(2n+1)} = \underset{n=0}{\frac{1}{1}} - \underset{n=1}{\frac{1}{3}} + \underset{n=2}{\frac{1}{2(5)}} - \underset{n=3}{\frac{1}{6(7)}} + \underset{n=4}{\frac{1}{24(9)}}$$

$0! = 1$ $\qquad = \boxed{\#} \simeq 0.747.$

Ex: Use power series to evaluate the following integrals:

a) $\int \frac{x^2}{3+5x^{15}}\,dx = \int \frac{x^2}{3}\cdot\frac{1}{1-\left(-\frac{5x^{15}}{3}\right)}\,dx = \frac{1}{3}\int x^2 \sum_{n=0}^{\infty}\left(-\frac{5x^{15}}{3}\right)^n dx$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1-A(x)} = \sum (A(x))^n$

$$= \frac{1}{3}\int x^2 \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 5^n\cdot x^{15n}}{3^n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 5^n}{3^{n+1}}\int x^{15n+2}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 5^n}{3^{n+1}}\cdot\frac{x^{15n+3}}{15n+3} + C$$

b) $\int x^2\sin(x^{17})\,dx = \int x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^{17})^{2n+1}\,dx$

$\sin x = \sum \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int x^{34n+17+2}\,dx.$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{x^{34n+20}}{34n+20} + C.$$

c) $\int x^e e^{\sqrt{x}}\,dx = \int x^e \sum_{n=0}^{\infty}\frac{(\sqrt{x})^n}{n!}\,dx$

$e^x = \sum \frac{x^n}{n!}$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int x^{\frac{n}{2}+e}\,dx.$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{x^{\frac{n}{2}+e+1}}{\frac{n}{2}+e+1} + C.$$

Multiplication and Division of Power Series:

Ex: Find the first three nonzero terms in the Maclaurin series for

a) $f(x) = e^x \sin x$

b) $f(x) = \tan x$

Note: A famous Euler's formula (Euler identity)

Prove the Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$