Section 11.11 ***The Binomial Series***

From algebra, how do we expand two – term – expression ➔ Pascal Triangle ➔ Binomial for positive integer exponent.

$(a+b)^0 =$ 1

$(a+b)^1 =$ $1a + 1b$

$(a+b)^2 =$ $1a^2 + 2ab + b^2$

$(a+b)^{-3.7} =$

$(a+b)^3 =$ $a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 =$ $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

⋮

Pascal's $\triangle$ ⟶

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

n choose K $\binom{n}{K}$

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!\cdot 7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7!}}{1\cdot 2\cdot 3\cdot \cancel{7!}} = 120$$

nCR.

One of ~~Newton~~ Taylor's accomplishments was to extend the Binomial Theorem to the case in which k is no longer a positive integer. In this case for $(a+b)^k$ is no longer a finite sum; it becomes an infinite series. Let's exam the Maclaurin series of $(1+x)^k$.

k is any real number.

$$f(x) = (1+x)^k = \sum ?? = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$n=0 \Rightarrow f(0) = (1+0)^k = 1$

$n=1 \Rightarrow f'(x) = k(1+x)^{k-1} \Rightarrow f'(0) = k$

$n=2 \Rightarrow f''(x) = k(k-1)(1+x)^{k-2} \Rightarrow f''(0) = k(k-1)$

$n=3 \Rightarrow f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \Rightarrow f'''(0) = k(k-1)(k-2).$

⋮

$n=n \Rightarrow f^{(n)}(0) = k(k-1)(k-2)\cdots(k-(n-1)) = k(k-1)(k-2)\cdots(k-n+1)$

$$f(x) = (1+x)^k = \sum^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n \quad \rightarrow \binom{k}{n}$$

$n=0$

The Binomial Series: If k is any real number and $|x|<1$, then $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$ where

$$\binom{k}{n} = \cancel{\frac{k!}{n!(k-n)!}} = \boxed{\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}} \text{ and } \binom{k}{0}=1$$

$n=1$ $\quad \frac{k-1+1}{1!} = k$

Binomial Series: $(1+x)^{\cancel{k}} = 1+\sum_{k=1}^{\infty}\binom{m}{k}x^k$ where we define $\binom{m}{1}=m;\ \binom{m}{2}=\frac{m(m-1)}{2!}$

And $\binom{m}{k}=\frac{m(m-1)(m-2)...(m-k+1)}{k!}$ for $k\geq 3$

Binomial: $\boxed{f(x)=(1+x)^k = 1+\sum_{n=1}^{\infty}\binom{k}{n}x^n}$ for k is any real number.

Ex: Using Binomial series to expand: $f(x)=\frac{1}{(1+x)^2}$

Sol:

$$f(x)=\frac{1}{(1+x)^2}=(1+x)^{-2} \quad \left\{ k=-2 \right.$$

$$f(x)=(1+x)^{-2}=1+\sum_{n=1}^{\infty}\binom{-2}{n}x^n.$$

$n=1 \Rightarrow \binom{-2}{1}=\frac{-2}{1!}=-2$

$n=2 \Rightarrow \binom{-2}{2}=\frac{-2(-2-1)}{2!}=\frac{2\cdot 3}{2}=3.$

$n=3 \Rightarrow \binom{-2}{3}=\frac{-2(-2-1)(-2-2)}{3!}=\frac{-\cancel{2}(-\cancel{3})(-4)}{1\cdot\cancel{2}\cdot\cancel{3}}=-4$

$n=4 \Rightarrow \binom{-2}{4}=\frac{-2(-2-1)(-2-2)(-2-3)}{4!}=\frac{-\cancel{2}(-\cancel{3})(-\cancel{4})(-5)}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}=5$

$$f(x)=\frac{1}{(1+x)^2}=\underset{x^0}{1}-2x+3x^2-4x^3+5x^5+\cdots$$

b) $f(x)=\sqrt{1+x}=(1+x)^{1/2}$ $\left\{ k=\frac{1}{2} \right.$

$$f(x)=\underset{n=0}{1}+\sum_{n=1}^{\infty}\binom{1/2}{n}x^{n}.$$

$\binom{k}{0}=1$

$\binom{k}{1}=k.$ $(k-4)$

$n=1 \Rightarrow \binom{1/2}{1}=\frac{1}{2}$

$n=2 \Rightarrow \binom{1/2}{2}=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}=-\frac{1}{8}$ $n=5$ $k(k-1)(k-2)$

$n=3 \Rightarrow \binom{1/2}{3}=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot \cancel{3}}=\frac{1}{16}.$ $\frac{\frac{-5}{2^4}}{2^3}$

$n=4 \Rightarrow \binom{1/2}{4}=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{\cancel{3}}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot \cancel{3}\cdot 4}=\frac{-5}{2^7}$

c) $\boxed{f(x)=\sqrt[3]{1+x}}$

$$f(x)=\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{2^7}x^4+\cdots$$

$f(x)=(1+x)^{1/3}$: $\left\{ k=\frac{1}{3} \right.$

$$f(x)=(1+x)^{1/3}=1+\sum_{n=1}^{\infty}\binom{1/3}{n}x^{n}.$$

$n=1 \Rightarrow \binom{1/3}{1}=\frac{1}{3}$;

$n=2 \Rightarrow \binom{1/3}{2}=\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}=\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\cdot \cancel{2}}=-\frac{1}{9}$

$n=3 \Rightarrow \binom{1/3}{3} = \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!} = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\cdot\cancel{2}\cdot 3} = \frac{5}{81}$

d) $f(x)=\frac{1}{\left(3+8x^3\right)^3}$

$n=4 \Rightarrow \binom{1/3}{4} = \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!} = \frac{\frac{1}{3}\left(-\frac{\cancel{2}}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{\cancel{8}}{3}\right)}{1\cdot\cancel{2}\cdot 3\cdot\cancel{4}}$

$= -\frac{10}{3^5} = -\frac{10}{403}$

$f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{403}x^4 + \cdots$

$f(x) = \frac{1}{\left(3+8x^3\right)^3}$

Section 11.12 Applications of Taylor Polynomials

Approximating Functions by polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

So, we let $T_n(x)$ be the first nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a.

$$T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i \text{ and let the error be } R_n(x)=\sum_{i=n+1}^{\infty}\frac{f^{(i)}(a)}{i!}(x-a)^i = \left|f(x)-T_n(x)\right|$$

We have from Taylor Inequality

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1} \text{ where } \left|f^{(n+1)}(x)\right| \le M$$

Ex: a) Approximate the function $f(x) = \sqrt[3]{x}$ by Taylor polynomial of degree 2 at $a = 8$

b) How accurate is this approximation when $7 \le x \le 9$?

Ex: The third Maclaurin polynomial for $\sin x$ is given by: $\sin x \approx x - \frac{x^3}{3!}$. Use Taylor's Theorem to approximate $\sin(0.1)$ by $T_3(0.1)$ and determine the accuracy of the approximation:

Ex: Determine the degree of the Taylor polynomial $T_n(x)$ expanded about $a = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

Ex: Approximate $\sin 2^{0}$ accurate to four decimal places.

Ex: a) What is the maximum error possible in using the approximation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ where $-0.3 \le x \le 0.3$? Use this approximation to find $\sin 12^0$ corrects to six decimal places?

b) For what values of x is this approximation accurate to within 0.00005?