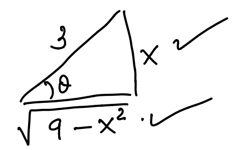
*Section 7.3* *Trigonometric Substitution*

Know this.

$$\begin{cases} a^2 - x^2 \Rightarrow x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta \\ \boxed{a^2 + x^2} \Rightarrow x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta \\ x^2 - a^2 \Rightarrow x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta \end{cases}$$

$\sin\theta = \frac{x}{3} = \frac{\text{Opp.}}{\text{Hyp.}}$




$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

Ex: Evaluate the following:

a) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{3^2-x^2}}{x^2}\,dx \to$ let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$.

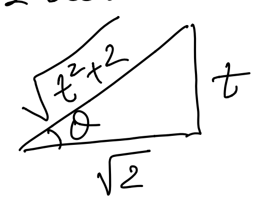
$$= \int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta} \cdot \cancel{3}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\,d\theta$$

$$= -\underbrace{\cot\theta}_{\text{"}\frac{\text{Adj}}{\text{Opp}}\text{"}} - \theta + C. = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

b) $\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{t^5}{\sqrt{t^2+(\sqrt{2})^2}}\,dt$. let $t = \sqrt{2}\tan\theta \Rightarrow dt = \sqrt{2}\sec^2\theta\, d\theta$.

$\tan\theta = \frac{t}{\sqrt{2}} = \frac{\text{Opp}}{\text{Adj}}$.




$$= \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{2\tan^2\theta + 2}} \cdot \sqrt{2}\sec^2\theta\, d\theta$$

$\sqrt{2(\tan^2\theta+1)} = \sqrt{2\sec^2\theta} = \sqrt{2}\cdot\sec\theta$.

$$= \int \frac{\sqrt{32}\cdot\tan^5\theta}{\cancel{\sqrt{2}}\cdot\cancel{\sec\theta}} \cdot \cancel{\sqrt{2}}\sec^{\cancel{2}}\theta\, d\theta = \sqrt{32}\int \tan^5\theta \cdot \sec\theta\, d\theta = \sqrt{32}\int \tan^4\theta \cdot \underbrace{\tan\theta\sec\theta\, d\theta}_{du} =$$

let $u = \sec\theta \Rightarrow du = \sec\theta\tan\theta\, d\theta$

$$= 4\sqrt{2}\int (\sec^2\theta - 1)^2 \cdot \sec\theta\tan\theta\, d\theta = 4\sqrt{2}\int (u^2-1)^2\,du = 4\sqrt{2}\int (u^4 - 2u^2 + 1)\,du.$$

$$= 4\sqrt{2}\left[\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right] + C = 4\sqrt{2}\left[\frac{1}{5}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^5 - \frac{2}{3}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^3 + \frac{\sqrt{t^2+2}}{\sqrt{2}}\right] + C.$$

c) $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{x^2\sqrt{(4x)^2-3^2}} \Rightarrow$ Let $4x = 3\sec\theta \rightarrow x = \frac{3}{4}\sec\theta$.

$4dx = 3\sec\theta\tan\theta\, d\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$.

$$= \int \frac{\frac{3}{4}\sec\theta\tan\theta\, d\theta}{\frac{9}{16}\sec^2\theta\sqrt{9\sec^2\theta - 9}}$$

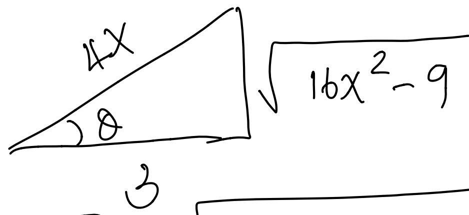
$\sqrt{9(\sec^2\theta-1)} = \sqrt{9\tan^2\theta} = 3\tan\theta$.

$$= \frac{3}{4}\cdot\frac{16}{9}\int \frac{\cancel{\sec\theta\tan\theta}}{\sec^2\theta \cdot 3\cancel{\tan\theta}}\, d\theta = \frac{4}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{4}{9}\int \cos\theta\, d\theta = \frac{4}{9}\sin\theta + C.$$

$$= \frac{4}{9}\cdot\frac{\sqrt{16x^2-9}}{4x} + C.$$

from $4x = 3\sec\theta$

$\sec\theta = \frac{4x}{3} = \frac{\text{Hyp}}{\text{Adj.}}$




d) $\displaystyle\int \sqrt{5+4x-x^2}\, dx \Rightarrow$ Note: $\sqrt{3\text{ terms}} \rightarrow$ completing square $= \boxed{\sqrt{2\text{ terms}}}$.

Complete square of: $5+4x-x^2 = 5 - \underbrace{(x^2-4x+4)}+4$

$= 9 - (x-2)^2$.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \{$

$$\int\sqrt{3^2-(x-2)^2}\, dx. = \int \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta.$$

$\sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$.

let $x-2 = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

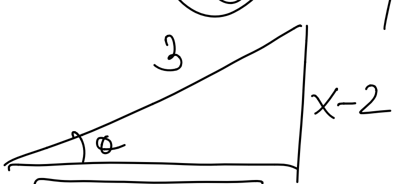
$$= \int 3\cos\theta\cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = 9\int \frac{1+\cos(2\theta)}{2}\, d\theta.$$

$$= \frac{9}{2}\int\left(1+\cos(2\theta)\right)d\theta = \frac{9}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C.$$

$$= \frac{9}{2}\left[\theta + \frac{1}{\cancel{2}}\cdot \cancel{2}\sin\theta\cos\theta\right] + C.$$

$\sin\theta = \frac{x-2}{3} = \frac{\text{Opp.}}{\text{Hyp.}}$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3}\right] + C$$



$\sqrt{9-(x-2)^2}$

**c)** $\displaystyle\int \frac{3x-5}{(21+12x-9x^2)^{3/2}}dx = \int \frac{(3x-2)-3}{[25-(3x-2)^2]^{3/2}}dx = \int \frac{5\sin\theta - 3}{[25-25\sin^2\theta]^{3/2}}\cdot \frac{5}{3}\cos\theta\, d\theta$

$[25(1-\sin^2\theta)]^{3/2} = [25\cos^2\theta]^{3/2} = \left[\sqrt{25\cos^2\theta}\right]^3 = 125\cos^3\theta$

$21+12x-9x^2 = 21-[(3x)^2-2(3x)\cdot 2+4]+4$
$= 25-(3x-2)^2$

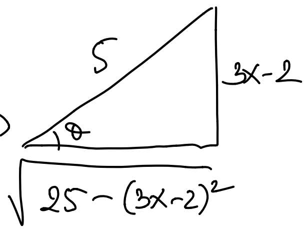
Let $3x-2 = 5\sin\theta$
$3dx = 5\cos\theta\, d\theta$
$dx = \frac{5}{3}\cos\theta\, d\theta$

$$= \int \frac{5\sin\theta-3}{125\cos^3\theta}\cdot\frac{5}{3}\cos\theta\, d\theta = \frac{1}{75}\int\frac{5\sin\theta-3}{\cos^2\theta}d\theta$$

$$= \frac{1}{75}\int\left(\underbrace{5\tan\theta\sec\theta} - 3\sec^2\theta\right)d\theta = \frac{1}{75}\left[5\sec\theta - 3\tan\theta\right]+C.$$

From $3x-2 = 5\sin\theta$

$\sin\theta = \frac{3x-2}{5} = \frac{\text{Opp.}}{\text{Hyp.}}$




$$= \frac{1}{75}\left[5\cdot\frac{5}{\sqrt{25-(3x-2)^2}} - 3\cdot\frac{3x-2}{\sqrt{25-(3x-2)^2}}\right] + C.$$

**f)** $\displaystyle\int\frac{dx}{\sqrt{x^2-6x+13}} = \int\frac{dx}{\sqrt{(x-3)^2+4}} = \int\frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta+4}}$

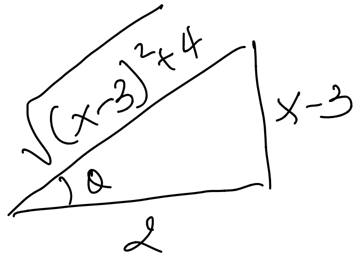
$x^2-6x+13 = \underbrace{x^2-6x+9}+4$
$= (x-3)^2+4$

$\sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$

Let $x-3 = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$.

$\int\sec^2\theta\, d\theta = \tan\theta + C$
$\int\sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|+C$

$$= \int\frac{\cancel{2}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{2}\sec\theta} = \int\sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|+C,$$




$$= \ln\left|\frac{\sqrt{(x-3)^2+4}}{2} + \frac{x-3}{2}\right| + C$$

from $x-3 = 2\tan\theta$

$\tan\theta = \frac{x-3}{2} = \frac{\text{Opp.}}{\text{Adj.}}$

g) $\displaystyle\int \frac{2x+4}{\left[-4x^2-12x-5\right]^{3/2}}\,dx$ ← Try this

h) $\displaystyle\int \frac{2x+3}{\left(4x^2-4x-3\right)^{5/2}}\,dx$