$$f(x)=\int \frac{2\,dx}{x+5}-\int\frac{7\,dx}{3x+1} = \frac{6x+2-7x-35}{(x+5)(3x+1)} = \int \frac{-x-33}{(x+5)(3x+1)}\,dx$$

Partial fraction

Ex: find partial fraction decomposition of $f(x)$.

$$f(x)=\frac{2x^2-x+1}{(x-1)\cdot(x+2)\cdot(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$$

Note: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+C$

$$A\Big|_{x=1} = \frac{2-1+1}{3(-2)} = -\frac{1}{3}$$

$$B\Big|_{x=-2} = \frac{8+2+1}{(-3)(-5)} = \frac{11}{15}$$

$$C\Big|_{x=3} = \frac{18-3+1}{2(5)} = \frac{8}{5}$$

$$f(x)=\frac{-1}{3(x-1)}+\frac{11}{15(x+2)}+\frac{8}{5(x-3)}$$

Ex: Evaluate the following:

a) $\int \frac{11x+5}{6x^2+x-2}dx = \int \frac{11x+5}{(2x-1)(3x+2)}dx = \int\left(\frac{A}{2x-1}+\frac{B}{3x+2}\right)dx.$

$$A\Big|_{x=\frac{1}{2}} = \frac{\frac{11}{2}+5}{\frac{3}{2}+2}\cdot\frac{2}{2} = \frac{11+10}{3+4} = \frac{21}{7}=3.$$

$$B\Big|_{x=-\frac{2}{3}} = \frac{-\frac{22}{3}+5}{-\frac{4}{3}-1}\cdot\frac{3}{3} = \frac{-22+15}{-4-3} = \frac{-7}{-7}=1.$$

← Algebra.

$$= \int \frac{3}{2x-1}dx + \int\frac{1}{3x+2}dx.$$

$$= \frac{3}{2}\ln|2x-1| + \frac{1}{3}\ln|3x+2| + C.$$

b) $\int \frac{43x-x^2-30}{x^3-2x^2-15x}dx = \int \frac{43x-x^2-30}{x(x+3)(x-5)}dx = \int\left(\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-5}\right)dx$

$$\begin{cases} A\Big|_{x=0} = \frac{-30}{3(-5)} = 2. \\ B\Big|_{x=-3} = \frac{43(-3)-9-30}{-3(-8)} = \frac{-168}{+24} = -7 \\ C\Big|_{x=5} = \frac{43(5)-25-30}{5(8)} = 4 \end{cases}$$

$\int\left(\frac{2}{x} - \frac{7}{x+3} + \frac{4}{x-5}\right)dx$

$= \underbrace{2\ln|x| - 7\ln|x+3| + 4\ln|x-5| + C}$

$= \ln\left|\frac{x^2\cdot(x-5)^4}{(x+3)^7}\right| + C.$

$\int \frac{23x^2+x+23}{(x+1)(5x^2+4)}dx = \int\left(\frac{A}{x+1} + \frac{Bx+C}{5x^2+4}\right)dx$

$\longrightarrow 23x^2+x+23 = A(5x^2+4) + (x+1)(Bx+C)$

$= (5A+B)x^2 + (C+B)x + 4A+C.$

$$\begin{cases} 5A+B = 23 \\ B+C = 1 \\ 4A+C = 23 \end{cases}$$

$A\Big|_{x=-1} = \frac{23-1+23}{9} = \frac{45}{9} = 5$

$25+B = 23 \Rightarrow B = -2$

$-2+C = 1 \Rightarrow C = 3.$

$= \int\left(\frac{5}{x+1} + \frac{-2x+3}{5x^2+4}\right)dx.$

$= 5\ln|x+1| - \frac{2}{10}\int\frac{10x}{5x^2+4}dx + 3\int\frac{1}{5x^2+4}dx.$ $\rightarrow \frac{3}{5}\int\frac{1}{x^2+\frac{4}{5}}dx$

$= 5\ln|x+1| - \frac{1}{5}\ln(5x^2+4) + \frac{3}{5}\int\frac{1}{\frac{4}{5}+x^2}dx$

$= 5\ln|x+1| - \frac{1}{5}\ln(5x^2+4) + \frac{3}{5}\cdot\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{\sqrt{5}x}{2}\right) + C$

$\int\frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

$a = \frac{2}{\sqrt{5}} \longrightarrow$ $\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

ex $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$\boxed{\deg(\text{top}) \geq \deg(\text{bot.}) \Rightarrow \text{Perform the long division b/4 partial fraction.}}$

$$\begin{array}{r} 2x \qquad\qquad\qquad\quad \\ x^2-2x-3 \overline{\big)\, 2x^3-4x^2-x-3} \\ 2x^3-4x^2-6x \quad\;\; \\ \hline 5x-3 \end{array} \leftarrow \text{Remainder}$$

$$= \int\left(2x + \frac{5x-3}{x^2-2x-3}\right)dx = x^2 + \underbrace{\int \frac{5x-3}{(x-3)(x+1)}\,dx}_{} \rightarrow \int\left(\frac{A}{x-3}+\frac{B}{x+1}\right)dx.$$

$$A\Big|_{x=3} = \frac{15-3}{4} = 3 \qquad \Rightarrow x^2 + \int\left(\frac{3}{x-3}+\frac{2}{x+1}\right)dx.$$

$$B\Big|_{x=-1} = \frac{-5-3}{-4} = 2 \qquad = x^2 + 3\ln|x-3| + 2\ln|x+1| + C,$$

ex $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = \int \frac{x^2}{(x-1)(x+1)^2}\,dx = \int\left[\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\right]dx.$

$$A\Big|_{x=1} = \frac{1}{2^2} = \frac{1}{4}$$

$$C\Big|_{x=-1} = \frac{1}{-2} = -\frac{1}{2}$$

plug $x=0 \Rightarrow 0 = -A + B + C$

$$0 = -\frac{1}{4} + B - \frac{1}{2}$$

$$B = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}.$$

$$\Rightarrow \frac{1}{4}\underbrace{\int \frac{1}{x-1}\,dx}_{} + \frac{3}{4}\underbrace{\int \frac{1}{x+1}\,dx}_{} - \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx.$$

let $u = x+1$
$du = dx$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| - \frac{1}{2}\underbrace{\int \frac{1}{u^2}\,du}_{} \rightarrow -\frac{1}{2}\int u^{-2}\,du$$

$$-\left(-\frac{1}{2}\right)\frac{u^{-1}}{-1}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\left(\frac{1}{x+1}\right) + C,$$

f) $\int \frac{e^{4t}+2e^{2t}-e^{t}}{e^{2t}+1}\,dt$

$= \int \frac{(e^{3t}+2e^{t}-1)e^{t}\,dt}{e^{2t}+1}$ $\left\{ \begin{array}{l} \text{let } u = e^{t} \\ du = e^{t}dt \end{array} \right.$

$= \int \frac{u^3+2u-1}{u^2+1}\,du$ ⇒ Long division: $u^2+1 \overline{\big) u^3+0u^2+2u-1}$ with quotient u; $-\;(u^3 \quad + u)$; $/ \quad / \quad u-1$ ⇐ Remainder.

$$= \int \left(u + \frac{u-1}{u^2+1}\right)du = \int \left(u + \frac{u}{u^2+1} - \frac{1}{u^2+1}\right)du$$

$$= \frac{1}{2}u^2 + \frac{1}{2}\ln(u^2+1) - \tan^{-1}(u) + C.$$

$$= \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^{t}) + C.$$

g) $\int \frac{\sin x\,dx}{\cos^2 x+\cos x-2}$ $\left\{ \begin{array}{l} \text{let } u = \cos x \\ du = -\sin x\,dx \\ -du = \sin x\,dx \end{array} \right.$

$$= \int \frac{-du}{u^2+u-2} = -\int \frac{1}{(u+2)(u-1)}\,du = -\int \left(\frac{A}{u+2} + \frac{B}{u-1}\right)du.$$

$$A\Big|_{u=-2} = \frac{1}{-3} = -\frac{1}{3} \quad ; \quad B\Big|_{u=1} = \frac{1}{3}.$$

$$= -\int \left(\frac{-1/3}{u+2} + \frac{1/3}{u-1}\right)du.$$

$$= -\left[-\frac{1}{3}\ln|u+2| + \frac{1}{3}\ln|u-1|\right] + C.$$

$$= \frac{1}{3}\left[\ln|\cos x+2| - \ln|\cos x-1|\right] + C.$$

Rationalizing Substitutions:

Ex: Evaluate the following:

a) $\int \frac{\sqrt{x+4}}{x}dx$ $\left\{ \begin{array}{l} \text{let } u=\sqrt{x+4} \;;\; u^2 = x+4 \\ 2u\,du = dx \;;\; x = u^2-4 \end{array} \right.$

$$= \int \frac{u}{u^2-4}\cdot 2u\,du = 2\int \frac{u^2-4+4}{u^2-4}du,$$

$$= 2\int\left[1+\frac{4}{(u-2)(u+2)}\right]du = 2\int\left[1+\frac{A}{u-2}+\frac{B}{u+2}\right]du.$$

$A\big|_{u=2} = \frac{4}{4} = 1$

$B\big|_{u=-2} = \frac{4}{-4} = -1$

$$= 2\int\left[1+\frac{1}{u-2}-\frac{1}{u+2}\right]du = 2\left[u+\ln|u-2|-\ln|u+2|\right]+C$$

$$= 2\left[\sqrt{x+4}+\ln\left|\sqrt{x+4}-2\right|-\ln\left|\sqrt{x+4}+2\right|\right]+C$$

b) $\cancel{\int \frac{1}{\sqrt[3]{x}+4\sqrt{x}}dx}$

$$\int \frac{1}{4\sqrt[3]{x}+\sqrt{x}}dx = \int \frac{1}{4\sqrt[3]{u^6}+\sqrt{u^6}}\cdot 6u^5du = 6\int \frac{u^5}{4u^2+u^3}du = 6\int \frac{u^{\cancel{5}\,3}}{\cancel{u^2}(4+u)}du$$

let $x = u^6$

$dx = 6u^5du$.

$u = x^{1/6}$

$$= 6\int \frac{u^3}{u^1+4}du. \leftarrow \boxed{\text{Synthetic Division ??}}$$

$$\begin{array}{r|rrrr} -4 & 1 & 0 & 0 & 0 \\ & & -4 & 16 & -64 \\ \hline & 1 & -4 & 16 & \boxed{-64} \end{array}$$

$$6\int\left(u^2-4u+16-\frac{64}{u+4}\right)du.$$

$$= 6\left[\frac{1}{3}u^3 - 2u^2 + 16u - 64\ln|u+4|\right] + C$$

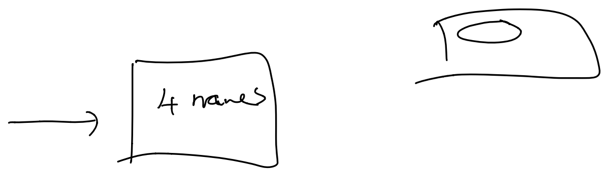
$$= 6\left[\frac{1}{3}\left(x^{1/6}\right)^3 - 2\left(x^{1/6}\right)^2 + 16x^{1/6} - 64\ln\left|x^{1/6}+4\right|\right] + C.$$

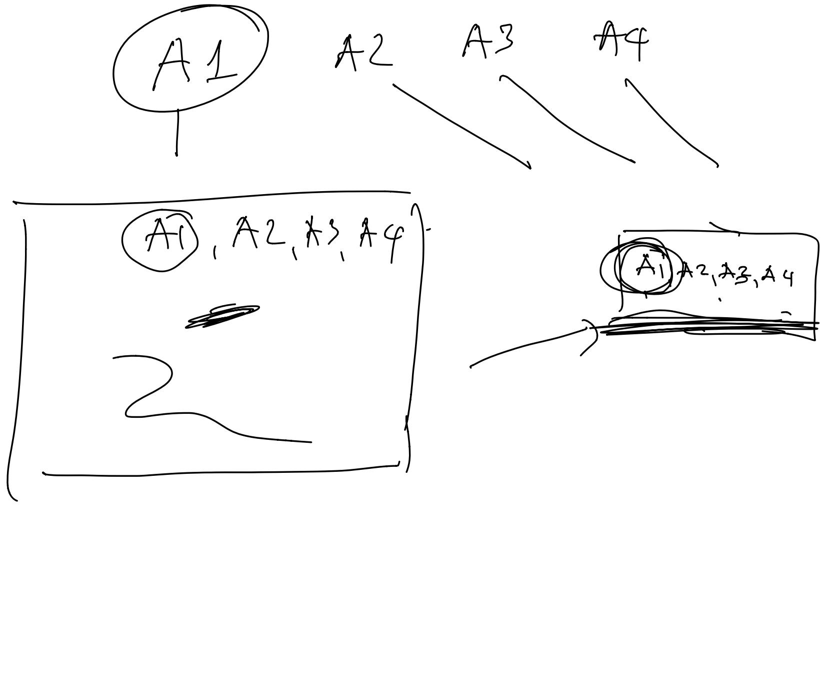
$$= 6\left[\frac{1}{3}\sqrt{x} - 2\sqrt[3]{x} + 16\sqrt[6]{x} - 64\ln\left|\sqrt[6]{x}+4\right|\right] + C$$

Quiz # 1. (Group quiz)

at most 4 people.

1) leader → scan, submit





1. Differentiate $f(x)$ (Without simplification)

$(uv)' = \ldots$

$\left(\frac{u}{v}\right)' = \ldots$

$(u \circ v)' = \ldots$

2. Find $\frac{dy}{dx}$ implicitly.

A ...

B ...

3. Integration:

a) u-sub.

b) by parts.

c) Trig. functions

d) Trig. sub.

e) partial fraction.

$\boxed{\text{FTC part II}}$