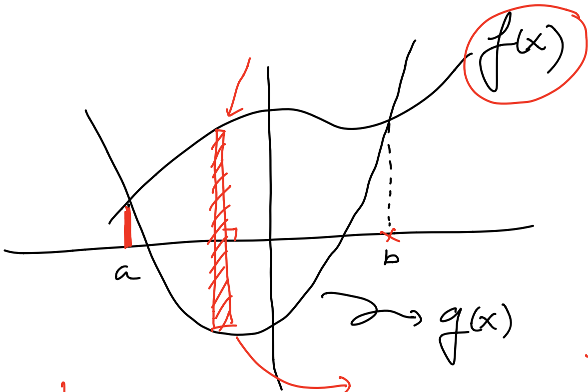
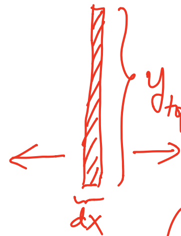
Section 6.1 ***Area between curves***

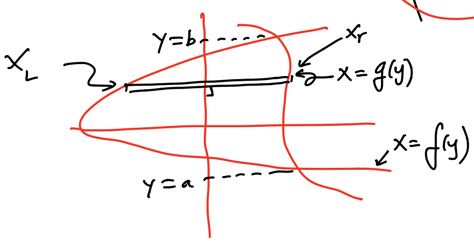
Given two functions $f(x)$ *and* $g(x)$ *over* $[a,b]$. How to find the area between the two curves:





$y_{top} - y_{bot} = f(x) - g(x)$.

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$



$L = x_{right} - x_{left} = g(y) - f(y)$

$$A = \int_a^b \big(g(y) - f(y)\big)\,dy$$

Ex: Find the area bounded between

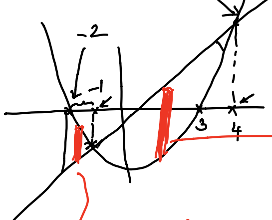
a) $y = x^2 - x - 6$ *and* $y = 2x - 2$ *for* ~~for~~ $-2 \le x \le 4$

$= (x-3)(x+2)$ → pts of intersections.

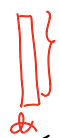
$x^2 - x - 6 = 2x - 2$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$ → $x = 4, -1$



$y_{top} - y_{bot} = x^2 - x - 6 - (2x-2)$
$= x^2 - 3x - 4$.



$y_{top} - y_{bot} = 2x - 2 - (x^2 - x - 6)$
$= 3x - x^2 + 4$.

$$\text{Area} = \int_{-2}^{-1} (x^2 - 3x - 4)\,dx + \int_{-1}^{4} (3x - x^2 + 4)\,dx$$

$$= \underbrace{\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x \Big|_{-2}^{-1}}_{\#} + \underbrace{\left(\frac{3}{2}x^2 - \frac{1}{3}x^3 + 4x\right)\Big|_{-1}^{4}}_{\#} = \text{Area}.$$

$+$

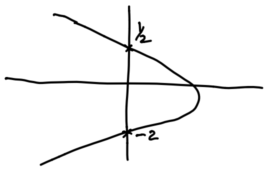
$=$

Sketch & Set-up the integrals for area.

b) $x=-2y^2-3y+2$ *and* $x=y-8$ *for* $-3\le y\le 2$

$= -(2y^2+3y-2)$

$= -(2y-1)(y+2)$



pts of intersection

$-2y^2-3y+2 = y-8$

$2y^2+4y-10=0$?

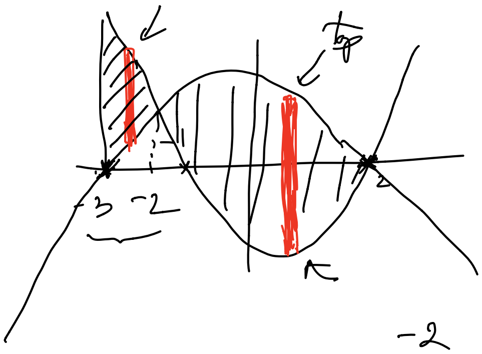
$y^2+2y-5=0$.

$(\quad)(\quad)$

c) $y=x^2-x-2$ *and* $y=-x^2-x+6$ *for* $\boxed{-3\le x\le 2}$

$=(x-2)(x+1)$ $\qquad = -(x^2+x-6)$

$= -(x+3)(x-2)$



pts of intersection: $\Rightarrow x^2 \cancel{-x} - 2 = -x^2 \cancel{-x} + 6$

$2x^2-8=0$

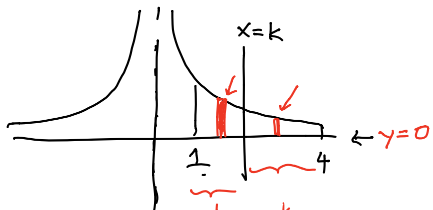
$x^2-4=0 \to x=\pm 2$.

$$\text{Area} = \int_{-3}^{-2}\left(x^2-x-2-(-x^2-x+6)\right)dx + \int_{-2}^{2}\left[-x^2-x+6-(x^2-x-2)\right]dx.$$

$$= \int_{-3}^{-2}(2x^2-8)\,dx + \int_{-2}^{2}(-2x^2+8)\,dx.$$

Ex: Find k so that $x = k$ bisects the area bounded by $y = \frac{1}{x^2}$ and $x - axis$ $1 \le x \le 4$

$y = \frac{1}{x^2}$



$$A = \int_1^k \left(\frac{1}{x^2} - 0\right) dx = \int_k^4 \left(\frac{1}{x^2} - 0\right) dx$$

$$\int_1^k x^{-2}\,dx = \int_k^4 x^{-2}\,dx.$$

$$-x^{-1}\Big|_1^k = -x^{-1}\Big|_k^4$$

$$-\left(\frac{1}{k} - 1\right) = -\left(\frac{1}{4} - \frac{1}{k}\right)$$

$$\frac{1}{k} - 1 = \frac{1}{4} - \frac{1}{k}$$

$$\frac{1}{k} + \frac{1}{k} = \frac{1}{4} + 1$$

$$\frac{2}{k} = \frac{5}{4}$$

$$\boxed{k = \frac{8}{5}}$$

Section 6.2 ***Volumes (Dish method, Cross Section)***

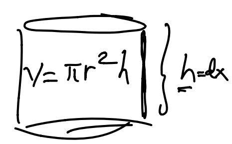
From volume for basic shapes such as box, cylinder, sphere… We can find volume of irregular shape but uniform height, and then we find volume by the area of the base times the height.

Def: Let S be a solid that lies between x = a and x = b. If the cross-sectional area of S in the plane P, through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the volume of S is

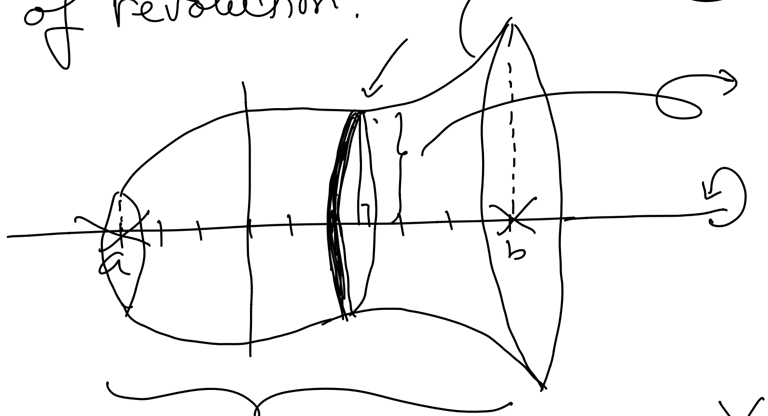
$$V = \lim_{n \to \infty} \sum A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

$y = f(x)$

cylinder



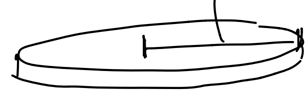
Volume of revolution:



$r = \;\; = y_{top} - y_{bot}$

$= f(x) - 0 = f(x)$

$r = f(x)$

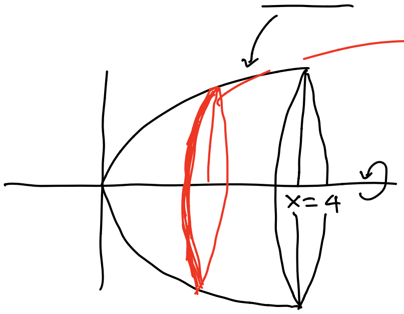


$$V = \pi r^2 dx$$

$$= \pi \cdot \int_a^b (f(x))^2 dx$$

Ex: Find the volume of the following region bounded by

a) $y = \sqrt{x}$, $y = 0$ and $x = 4$, rotated about the x – axis.



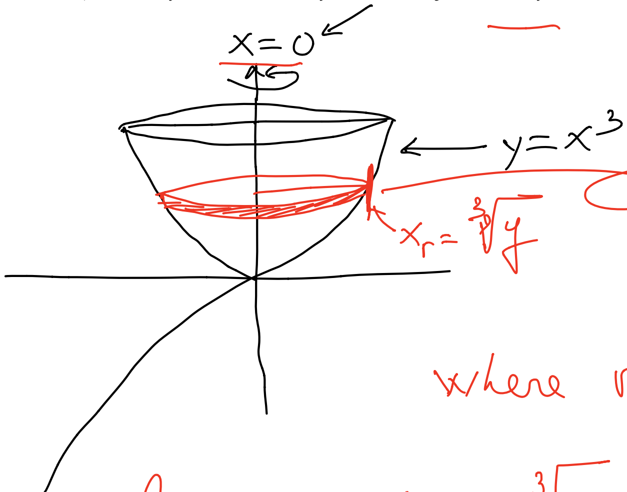
$V = \pi r^2 dx$

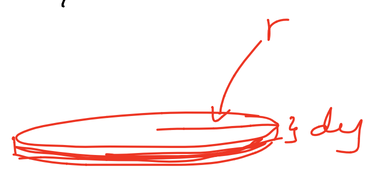
where : $r = \quad = y_{top} - y_{bot} = \sqrt{x} - 0 = \sqrt{x}$ ($y=\sqrt{x}$, $y=0$)

$$V = \pi \int_0^4 (\sqrt{x})^2 dx .$$

$$= \pi \int_0^4 x\,dx = \frac{\pi}{2} x^2 \Big|_0^4 = \boxed{8\pi}$$

b) $y = x^3$ and $y - axis$ for $0 \le y \le 1$; about the y-axis.





$V = \pi \cdot r^2 \cdot dy$.

where r is $= x_{right} - x_{left}$ $= x^3 = 0$

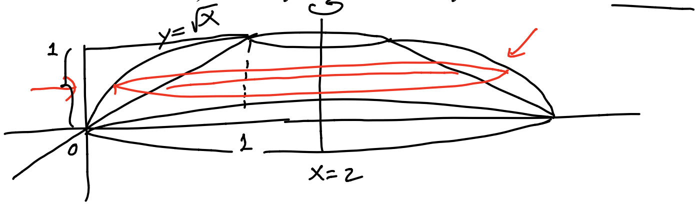
$= \sqrt[3]{y} - 0 = \sqrt[3]{y}$

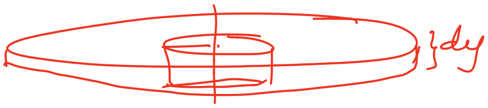
from $y = x^3 \Rightarrow x = \sqrt[3]{y}$

Volume: $V = \pi \int_0^1 (\sqrt[3]{y})^2 dy = \pi \int_0^1 y^{2/3} dy$

$$= \pi \cdot \frac{3}{5} y^{5/3} \Big|_0^1 = \boxed{\frac{3\pi}{5}}$$

c) $y=\sqrt{x}$ and $y=x$ about $x=2$





$V = \pi(R_o^2 - r_i^2)\,h.$

$= \pi(R_o^2 - r_i^2)\,dy.$

Where $R_o =$ ($y=\sqrt{x}$, $x=y^2$ to $x=2$) $= x_r - x_L = 2-y^2$

$r_i =$ ($y=x$, $x=y$ to $x=2$) $= x_r - x_L = 2-y$

$$V = \int_0^1 \pi\left[(2-y^2)^2 - (2-y)^2\right] dy$$

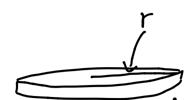
Washer.

$V = $  $-$ 

$= \pi R_o^2 h - \pi r_i^2 h.$

$= \pi(R_o^2 - r_i^2)\,h.$

Solid disc:



dx : h x-axis

dy : h y-axis

$V = \pi \cdot r^2 \cdot \{dx \text{ or } dy\}.$

Washer disc:

R_o : outter radius

r_i : Inner "

$V = \pi(R_o^2 - r_i^2)$ dx or dy

d) $y = x^2 - x - 6$ and $y = 2x+4$ about $y=14$

$= (x-3)(x+2)$

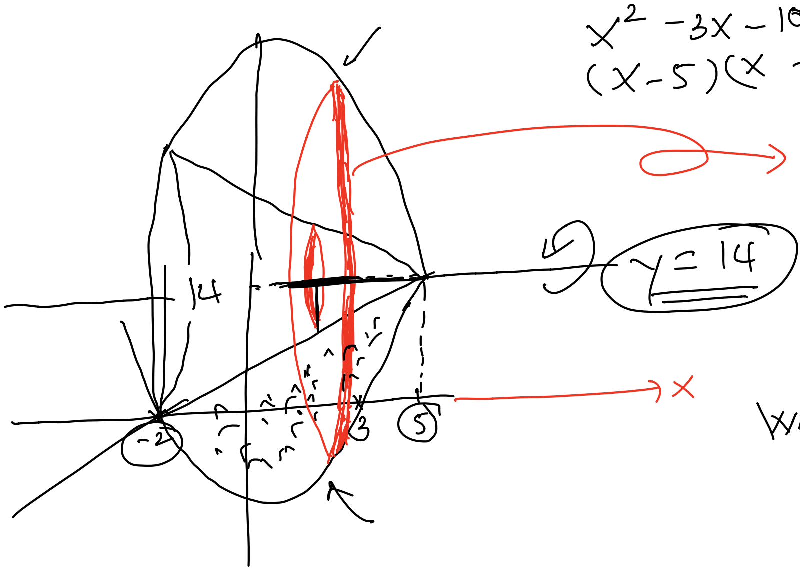
$25-5-6$

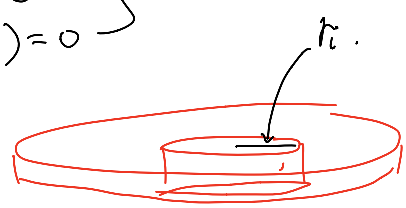
pts of intersection:

$x^2 - x - 6 = 2x+4$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$ $\rightarrow x = 5, -2$





$V = \pi(R_o^2 - r_i^2)\,dx$

Where R_o: ($Y_{top} = 14$, $Y_{bot} = x^2 - x - 6$) $= y_{top} - y_{bot} = 14 - (x^2 - x - 6)$

$= 20 - x^2 + x$

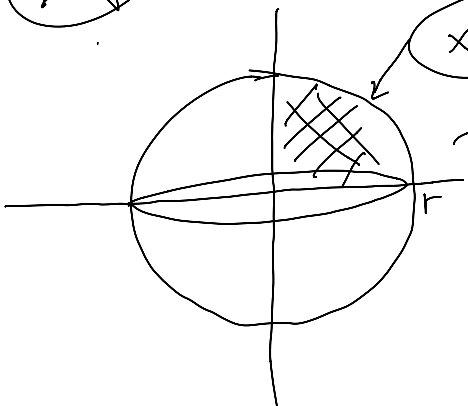
r_i: ($Y_{top} = 14$, $Y_{bot} = 2x+4$) $= Y_{top} - Y_{bot} = 14 - (2x+4)$

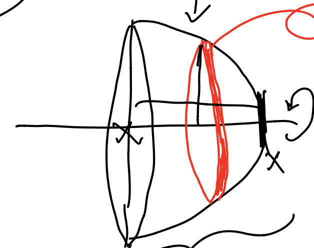
$= 10 - 2x.$

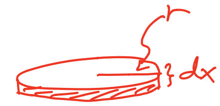
$$V = \int_{-2}^{5} \pi\left((20 - x^2 + x)^2 - (10-2x)^2\right) dx$$

Ex: Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

$x^2 + y^2 = r^2$







$V = \pi \cdot r^2\,dx$

$y_{top} = \sqrt{r^2 - x^2}$

where r:

$y_{bot} = 0$.

$y = \pm\sqrt{r^2 - x^2}$.

$r = y_{top} - y_{bot}$

$= \sqrt{r^2 - x^2} - 0 = \sqrt{r^2 - x^2}$

$$V = 2\pi \int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx = 2\pi \int_0^r (r^2 - x^2)\,dx = 2\pi \left[r^2 x - \frac{1}{3}x^3 \right]_0^r$$

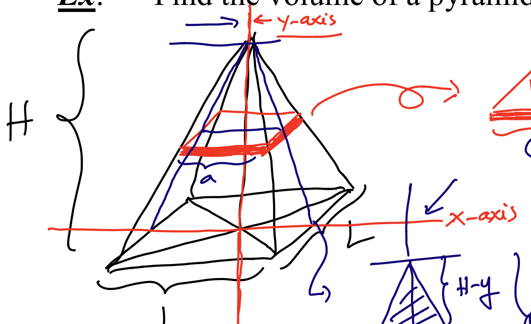
$$= 2\pi \left[r^3 - \frac{1}{3}r^3 \right] = 2\pi \cdot \frac{2}{3} r^3$$

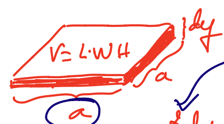
$$= \boxed{\frac{4\pi}{3} r^3} = V$$

double the volume.

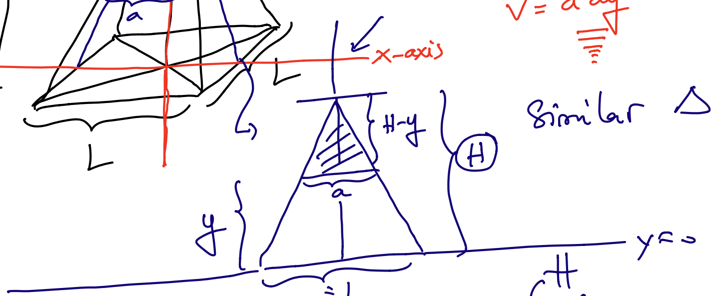
Parallel Cross Sections

Ex: Find the volume of a pyramid whose base is a square with side L and whose height is H.



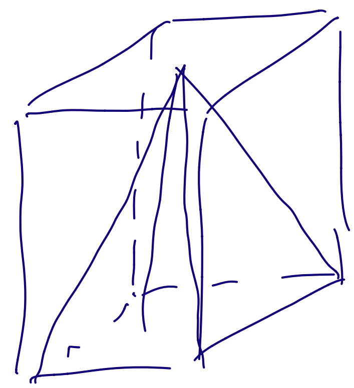


$V = a^2 dy$



Similar Δ: $\frac{H-y}{H} = \frac{a}{L}$.

$a = \frac{L}{H}(H-y)$



$$V = \int_0^H \left(\frac{L}{H}(H-y) \right)^2 dy$$

$$= \frac{L^2}{H^2} \int_0^H (H^2 - 2Hy + y^2)\,dy.$$

$$= \frac{L^2}{H^2} \left[H^2 y - \frac{\cancel{2}Hy^2}{\cancel{2}} + \frac{1}{3}y^3 \right]_0^H = \frac{L^2}{\cancel{H^2}} \left[\cancel{H^3} - \cancel{H^3} + \frac{1}{3}H^3 \right]$$

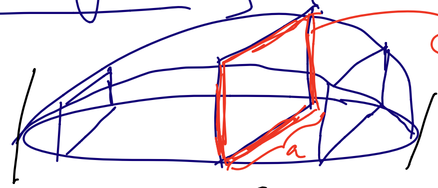
$$= \boxed{\frac{1}{3}L^2 H}$$

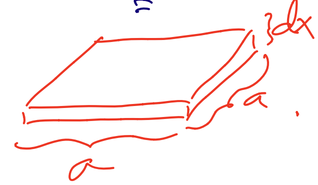
base

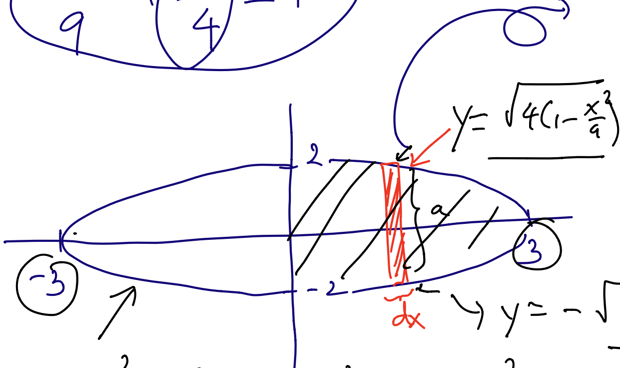
Ex: The base of a solid is bounded by $\{(x,y) | 4x^2+9y^2 \le 36\}$. All parallel cross sections are squares which perpendicular to the base and the x – axis. Find the volume of the solid.

$$B = \{(x,y) \mid 4x^2+9y^2 \le 36\} = \{(x,y) \mid \frac{x^2}{9}+\frac{y^2}{4} \le 1\}$$

$$\frac{x^2}{9}+\frac{y^2}{4} \le 1$$







$$V = a^2\,dx$$

where $a := y_{top} - y_{bot} = \sqrt{4(1-\frac{x^2}{9})} + \sqrt{4(1-\frac{x^2}{9})}$

$$V = \int_{-3}^{3}\left(2\sqrt{4\left(1-\frac{x^2}{9}\right)}\right)^2 dx = \int_{-3}^{3} 16\left(1-\frac{x^2}{9}\right) dx$$

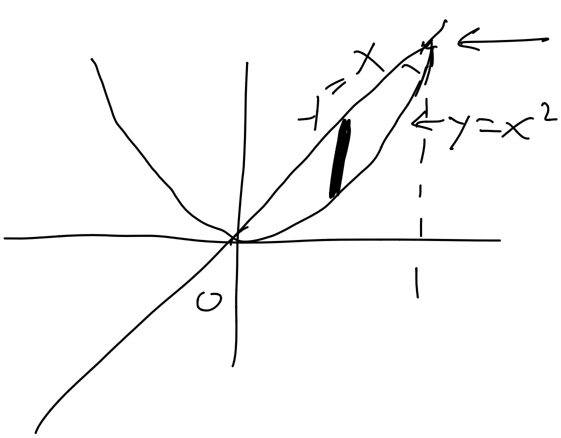
$$\frac{x^2}{9}+\frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1-\frac{x^2}{9}$$

$$y = \pm\sqrt{4\left(1-\frac{x^2}{9}\right)}$$

$$V = 2\int_0^3 16\left(1-\frac{x^2}{9}\right) dx = 32\left[x - \frac{1}{27}x^3\right]_0^3$$

$$= 32[3-1] = \boxed{64}$$

Ex: The base of a solid is bounded by $y = x^2$ *and* $y = x$. All parallel cross sections are semi-circle perpendicular to the base and the x-axis.



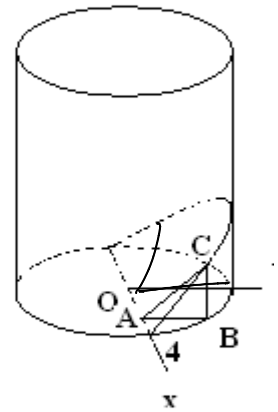
$$V = \left(\frac{1}{2}\pi r^2\right) \cdot dx$$

where $r = \frac{1}{2}(\quad) = \frac{1}{2}(x - x^2)$

$$V = \int_0^1 \frac{1}{2}\pi\left(\frac{1}{2}(x-x^2)\right)^2 dx$$

$$= \frac{\pi}{8}\int_0^1 (x^2 - 2x^3 + x^4)\,dx = \boxed{\frac{\pi}{8}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]}$$

Ex: A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30 degree along the diameter of the cylinder. Find the volume of the wedge.

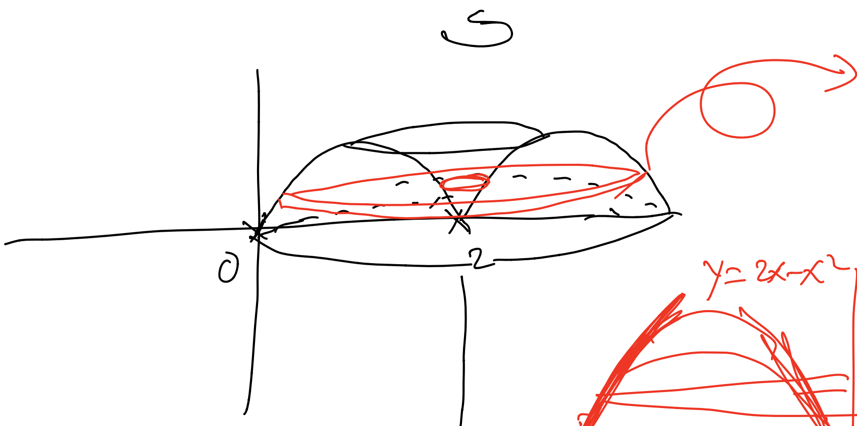


?

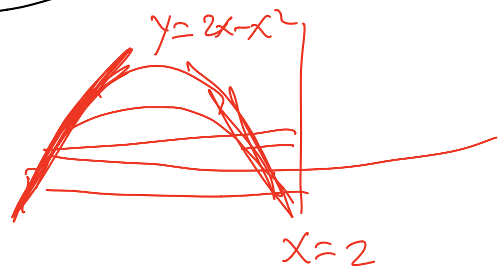
6.3 ***Volume by cylindrical shell method.***

The region bounded by $y = 2x - x^2$, $y = 0$ } rotate about $\underline{\underline{x = 2}}$. $\Rightarrow$ Find its volume.

$= x(2-x)$



$V = \pi(R_o^2 - r_i^2)$ { dx or dy



Ex The region bounded by the following curves, $\boxed{\text{sketch}}$ and $\boxed{\text{set up}}$ integrals for volume:

a) $\cancel{Y} = 6 + x - x^2$ *and* $y = 2x - 6$

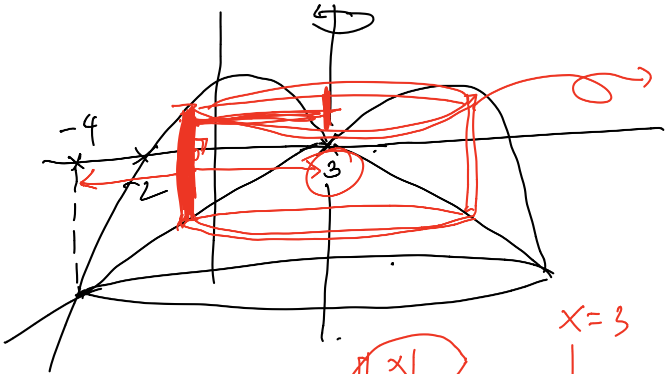
i) rotates about the line $x = 3$

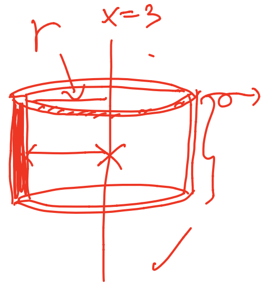
$$y = 6 + x - x^2 = -(x^2 - x - 6) = (-)(x-3)(x+2)$$

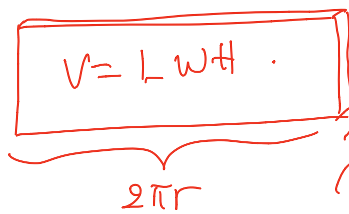
pts of intersection: $6 + x - x^2 = 2x - 6 \rightarrow x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x = -4, 3.$





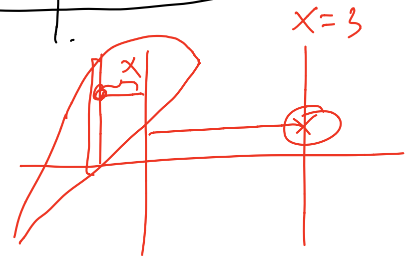


$\} \, ? = y_{top} - y_{bot}$

$= 6 + x - x^2 - (2x - 6)$

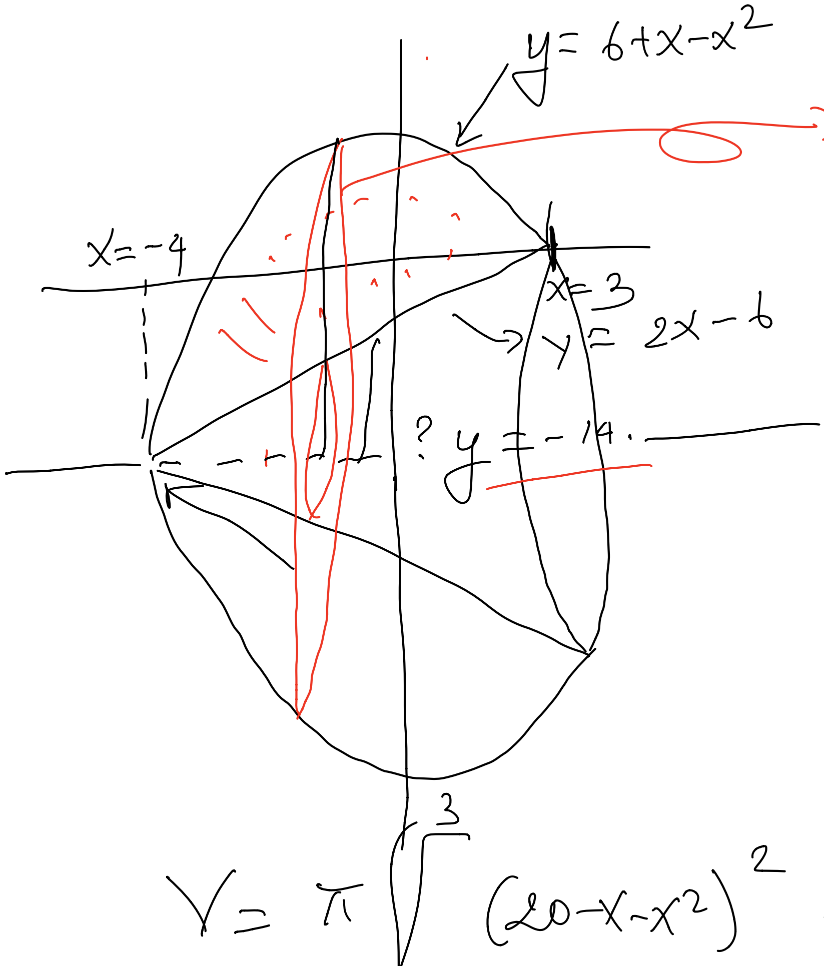
$= 12 - x - x^2$

$2\pi r = 2\pi(3 - x)$



$$V = \int_{-4}^{3} \underbrace{2\pi(3-x)}_{L} \underbrace{(12 - x - x^2)}_{W} \underbrace{dx}_{H}$$

ii) rotates about the line $y = -14$



$$V = \pi \cdot (R_o^2 - r_i^2)\, dx$$

Where $R_o = y_{top} - y_{bot}$, $y_{top} = 6 + x - x^2$, $y_{bot} = -14$

$$R_o = (6 + x - x^2) - (-14)$$

$$= 20 + x - x^2.$$

$r_i = y_{top} - y_{bot}$, $y_{top} = 2x - 6$, $y_{bot} = -14.$ $\} \; 2x - 6 + 14 = 2x + 8.$

$$V = \pi \int_{-4}^{3} \left[(20 - x - x^2)^2 - (2x + 8)^2 \right] dx.$$

b) The region bounded by $x = -y^2 + 2y + 3$ and $x + y = -1 \Rightarrow x = -1-y$

i) rotated about the line $y = 4$

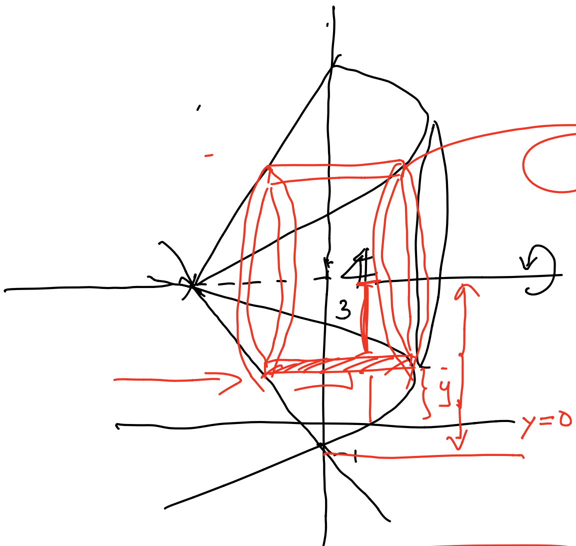
$x = -(y^2 - 2y - 3) = -(y+1)(y-3)$

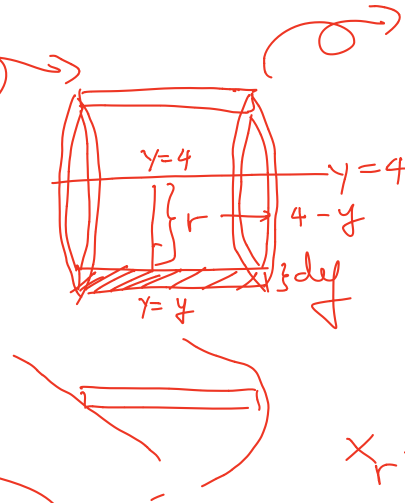
pts of intersections

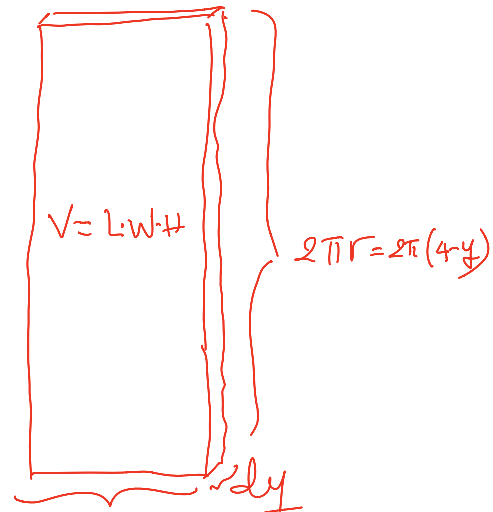
$-y^2 + 2y + 3 = -1 - y$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0 \rightarrow y = 4, -1$





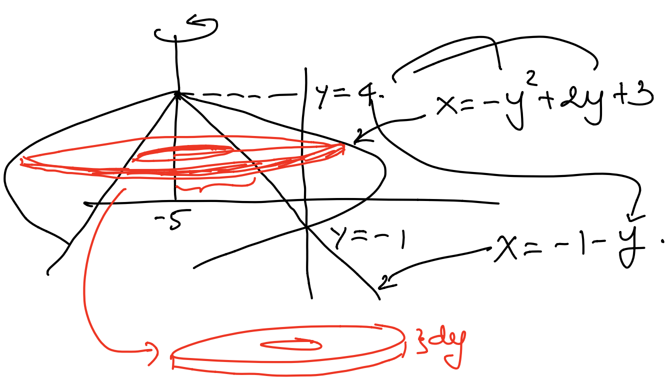


$x_r - x = -y^2 + 2y + 3 - (-1-y)$

$= -y^2 + 3y + 4$

$$V = \int_{-1}^{4} 2\pi(4-y)(4+3y-y^2)\,dy$$

ii) rotated about the line $x = -5$



$V = \pi(R_o^2 - r_i^2)\,dy$

where $R_o := \; = x_r - x_L = -y^2 + 2y + 3 - (-5)$

$= -y^2 + 2y + 8$

$x = -5$

$r_i := \; = x_r - x_L = (-1-y) - (-5)$

$= 4 - y$

$$V = \pi \int_{-1}^{4} \left[(-y+2y-8)^2 - (4-y)^2\right] dy$$