Section 7.8 *Improper Integrals*

Ex: Evaluate the integral: $\int_0^2 \frac{1}{(x-1)^2}\,dx =$

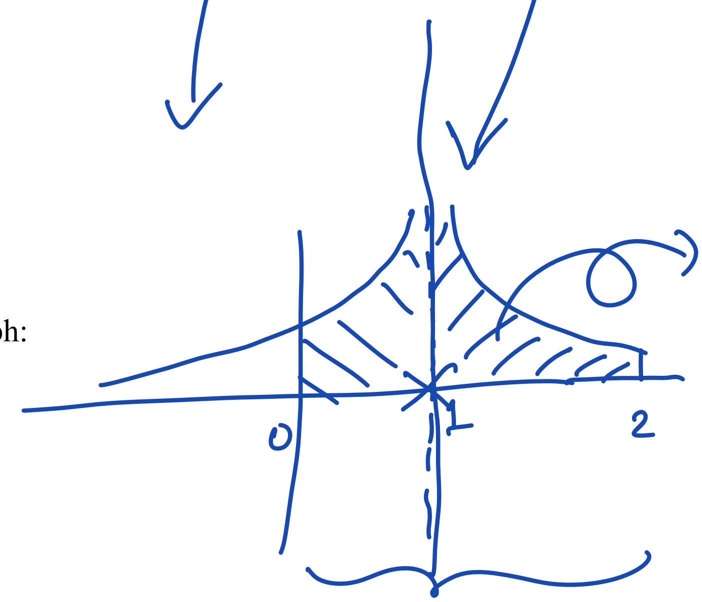
$$\begin{cases} \text{let } u = x-1 \\ du = dx \end{cases} \qquad \begin{cases} x=2 \Rightarrow u=1 \\ x=0 \Rightarrow u=-1 \end{cases}$$

$$= \int_{-1}^{1} \frac{1}{u^2}\,du = \int_{-1}^{1} u^{-2}\,du = \frac{u^{-1}}{-1}\Big|_{-1}^{1} = -\left[\frac{1}{u}\right]_{-1}^{1}$$

$$= -\left[1+1\right] = -2.$$

$$y = f(x) = \frac{1}{(x-1)^2}$$

Let's check the graph:

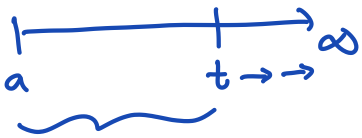


Area > 0,

Type 1: Improper Integrals (Infinite Limits of Integration)

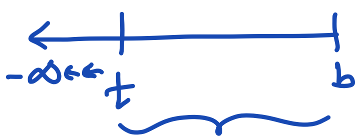
1. If f is continuous on the interval $[a,\infty)$

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$



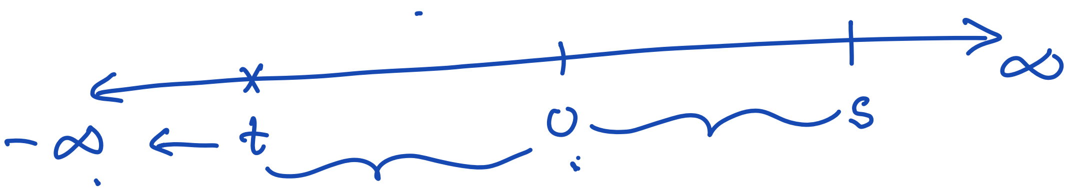
2. If f is continuous on the interval $(-\infty,b]$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$



3. If f is continuous on the interval $(-\infty,\infty)$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$



$$= \lim_{t\to-\infty}\int_t^0 f(x)\,dx + \lim_{s\to\infty}\int_0^s f(x)\,dx$$

Ex: Evaluating the following improper integrals:

a) $\int_0^{\infty} \frac{x}{\left(4x^2+1\right)^{5/2}}dx = \lim_{t\to\infty}\int_0^t \frac{x}{(4x^2+1)^{5/2}}dx$

$$\left\{\begin{aligned} \text{let } u &= 4x^2+1 \\ du &= 8x\,dx \\ \frac{du}{8} &= x\,dx. \end{aligned}\right.$$

$$= \lim_{t\to\infty}\int \frac{du/8}{u^{5/2}} = \frac{1}{8}\lim_{t\to\infty}\int u^{-5/2}\,du.$$

$$= \frac{1}{8}\lim_{t\to\infty} u^{-3/2}\cdot\left(-\frac{2}{3}\right) = -\frac{1}{12}\lim_{t\to\infty}\left(4x^2+1\right)^{-3/2}\Big|_0^t$$

$$= -\frac{1}{12}\lim_{t\to\infty}\cdot\left[\underbrace{\frac{1}{(4t^2+1)^{3/2}}}_{=0} - 1\right] = -\frac{1}{12}(-1) = \frac{1}{12}\ \{\text{convergent}\}$$

b) $\int_1^{\infty} \frac{x^2}{\sqrt[3]{7x^3+1}}dx = \lim_{t\to\infty}\int_1^t \frac{x^2}{\sqrt[3]{7x^3+1}}dx$

$$\left\{\begin{aligned} \text{let } u &= \sqrt[3]{7x^3+1} \\ u^3 &= 7x^3+1 \\ 3u^2\,du &= 21x^2\,dx \\ \frac{1}{7}u^2\,du &= x^2\,dx. \end{aligned}\right.$$

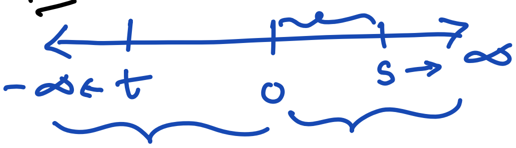
$$= \lim_{t\to\infty}\int \frac{\frac{1}{7}u^{\cancel{2}}\,du}{\cancel{u}} = \frac{1}{7}\lim_{t\to\infty}\int u\,du.$$

$$= \frac{1}{7}\lim_{t\to\infty}\frac{1}{2}u^2 = \frac{1}{14}\lim_{t\to\infty}\left(\sqrt[3]{7x^3+1}\right)^2\Big|_1^t.$$

$$= \frac{1}{14}\lim_{t\to\infty}\left[\left(\sqrt[3]{7t^3+1}\right)^2 - 4\right] = \infty\ \{\text{divergent}\}$$

c) $\int_{-\infty}^{\infty} \frac{x}{x^4+9}dx = \lim_{t\to-\infty}\int_{t}^{0}\frac{x}{x^4+9}dx + \lim_{s\to\infty}\int_{0}^{s}\frac{x}{x^4+9}dx$

$\left\{\begin{array}{l} \text{let } u = x^2 \\ du = 2x\,dx \\ \frac{du}{2} = x\,dx \end{array}\right.$



$= \lim_{t\to-\infty}\int \frac{\frac{du}{2}}{u^2+9} + \lim_{s\to\infty}\int \frac{\frac{du}{2}}{u^2+9}\,du$

$= \frac{1}{2}\left[\lim_{t\to-\infty}\int\frac{du}{u^2+9} + \lim_{s\to\infty}\int\frac{du}{u^2+9}\right]$

d) $\int_{-\infty}^{0} \frac{x}{e^{1-3x^2}}dx$

$= \frac{1}{2}\left[\lim_{t\to-\infty} \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + \lim_{s\to\infty}\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]$

$= \frac{1}{6}\left[\lim_{t\to-\infty}\tan^{-1}\left(\frac{x^2}{3}\right)\Big|_{t}^{0} + \lim_{s\to\infty}\tan^{-1}\left(\frac{x^2}{3}\right)\Big|_{0}^{s}\right]$

$= \frac{1}{6}\left[\lim_{t\to-\infty}\left[0 - \tan^{-1}\left(\frac{t^2}{3}\right)\right] + \lim_{s\to\infty}\left[\tan^{-1}\left(\frac{s^2}{3}\right) - 0\right]\right]$

$= \frac{1}{6}\left[-\frac{\pi}{2} + \frac{\pi}{2}\right] = 0$ {Convergent}

$$\begin{cases}\int \frac{1}{x^1}dx = \ln|x|+C \\ \int \frac{1}{x^{p\neq 1}}dx = \int x^{-p}dx = \frac{x^{-p+1}}{-p+1}+C\end{cases}$$

P-Test Theorem: For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent? ← finite #.

Proof:

If $p=1 \Rightarrow \int_1^\infty \frac{1}{x^1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big|_1^t$

$$= \lim_{t\to\infty}\left(\ln|t| - \underbrace{\ln|1|}_{0}\right) = \ln|\infty| = \infty$$

$p=1 \Rightarrow \int_1^\infty \frac{1}{x^p}dx$ is divergent

If $p\neq 1 \Rightarrow \int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t$

$$= \lim_{t\to\infty}\left(\frac{t^{-p+1} - 1^{-p+1}}{-p+1}\right) = \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p} - 1\right)$$

$\Rightarrow$ For convergence $\Rightarrow 1-p<0 \Rightarrow \boxed{p>1}$

P-Test Thm: $\int_{a>0}^\infty \frac{1}{x^p}dx$ is $\begin{cases}\text{convergent if } p>1 \\ \text{divergent if } p\leq 1\end{cases}$

Ex: Test for convergence / divergence:

a) $\int_1^\infty \frac{\sqrt{x^5}}{x^3}dx = \int_1^\infty \frac{1}{x^p}dx$

$\Rightarrow \int_1^\infty \frac{x^{5/2}}{x^3}dx = \int_1^\infty \frac{1}{x^{3-\frac{5}{2}}}dx$

$= \int_1^\infty \frac{1}{x^{1/2}}dx \rightarrow p=\frac{1}{2}<1$

By P-Test: $\int_1^\infty f(x)dx$ is divergent.

b) $\int_1^\infty \frac{x}{\sqrt[3]{x^8}}dx = \int_1^\infty \frac{1}{x^p}dx$

$\int_1^\infty \frac{x}{x^{8/3}}dx = \int_1^\infty \frac{1}{x^{\frac{8}{3}-1}}dx$

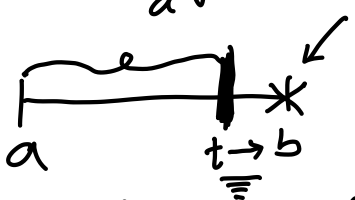
$= \int_1^\infty \frac{1}{x^{5/3}}dx \rightarrow p=\frac{5}{3}>1$

By P-Test $\Rightarrow \int_1^\infty f(x)dx$ is convergent

Type 2: ***Discontinuous Integrands***

1. If f is continuous on the interval [a, b) and approaches infinity at b.

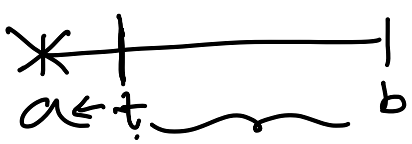
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$



$f(b) =$ undefined.

2. If f is continuous on the interval (a, b] and approaches infinity at a.

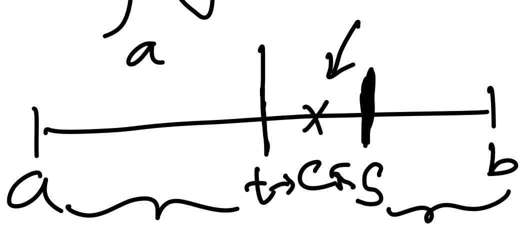
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \quad \checkmark$$



$f(a) =$ undefined

3. If f is continuous on the interval [a,b], except for some c in (a, b).

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{s \to c^+} \int_s^b f(x)\,dx.$$



$f(c) =$ undefined

Ex: Evaluate the following improper integrals

a) $\int_0^2 \frac{x^2}{\sqrt[3]{x^3-8}}dx = \lim_{t\to 2^-}\int_0^t \frac{x^2}{\sqrt[3]{x^3-8}}dx$

$8-8$

$0 \quad t\to 2$

$$\begin{cases} u = \sqrt[3]{x^3-8} \\ u^3 = x^3-8 \\ 3u^2du = 3x^2dx \\ u^2du = x^2dx \end{cases}$$

$$= \lim_{t\to 2^-}\int \frac{u^{\cancel{2}}du}{\cancel{u}} = \lim_{t\to 2^-} \frac{u^2}{2}$$

$2^- \quad -2$

$$= \frac{1}{2}\lim_{t\to 2^-}\left(\sqrt[3]{x^3-8}\right)^2\Big|_0^t = \frac{1}{2}\lim_{t\to 2^-}\left(\left(\sqrt[3]{t^3-8}\right)^2 - 4\right)$$

$\to 0$

$$= \boxed{-2} \Rightarrow \text{"Convergent"}$$

b) $\int_{2/3}^1 \frac{1}{(\sqrt[3]{3x-2})^4}dx = \lim_{t\to \frac{2}{3}^+}\int_t^1 \frac{1}{(\sqrt[3]{3x-2})^4}dx$

$$\begin{cases} \text{Let } u = \sqrt[3]{3x-2} \\ u^3 = 3x-2 \\ 3u^2du = 3dx \\ u^2du = dx. \end{cases}$$

$\frac{2}{3} \leftarrow t \quad 1$

$$= \lim_{t\to \frac{2}{3}^+}\int \frac{1}{u^{\cancel{4}\,2}} \cdot \cancel{u^2}du. = \lim_{t\to \frac{2}{3}^+}\int u^{-2}du.$$

$$= \lim_{t\to \frac{2}{3}^+} \frac{u^{-1}}{-1} = -\lim_{t\to \frac{2}{3}} \frac{1}{\sqrt[3]{3x-2}}\Big|_t^1$$

$$= -\lim_{t\to \frac{2}{3}^+}\left[1 - \frac{1}{\sqrt[3]{3t-2}}\right] = \infty \quad \text{"Divergent"}$$

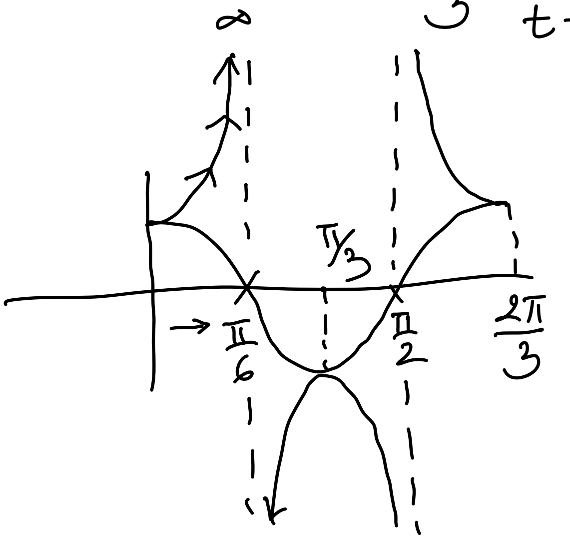
$\int \tan(3x)dx = \frac{1}{3}\ln|\sec(3x)| + C$

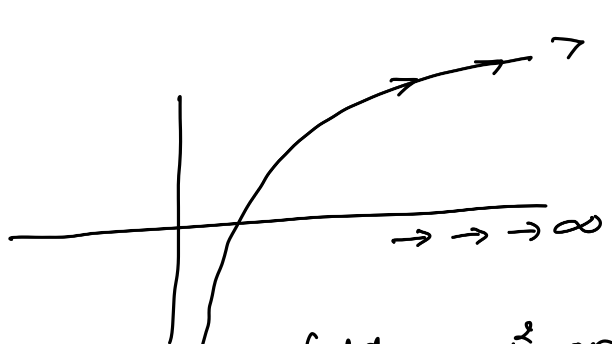
c) $\int_0^{\frac{\pi}{6}} \tan(3x)\,dx = \lim_{t\to\frac{\pi}{6}^-}\int_0^t \tan(3x)\,dx = \lim_{t\to\frac{\pi}{6}^-} \frac{1}{3}\ln|\sec(3x)|\Big|_0^t$

$$= \frac{1}{3}\lim_{t\to\frac{\pi}{6}^-}\Big[\underbrace{\ln|\sec(3t)|}_{\infty} - \underbrace{\ln|\underbrace{\sec(0)}_{=1}|}_{\ln 1 = 0.}\Big] = \infty$$

∞ DNE

"Divergent"





d) $\int_3^6 \frac{x}{x^2-25}\,dx = \lim_{t\to 5^-}\int_3^t \frac{x}{x^2-25}\,dx + \lim_{s\to 5^+}\int_s^6 \frac{x}{x^2-25}\,dx$

let $u = x^2-25$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \lim_{t\to 5^-}\int \frac{\frac{du}{2}}{u} + \lim_{s\to 5^+}\int \frac{\frac{du}{2}}{u}.$$

$$= \frac{1}{2}\left[\lim_{t\to 5^-}\int \frac{1}{u}\,du + \lim_{s\to 5^+}\int \frac{1}{u}\,du\right]$$

$$= \frac{1}{2}\left[\lim_{t\to 5^-} \ln|x^2-25|\Big|_3^t + \lim_{s\to 5^+} \ln|x^2-25|\Big|_s^6\right]$$

$$= \frac{1}{2}\Big[\underbrace{\lim_{t\to 5^-}\Big(\ln|t^2-25| - \ln|9-25|\Big)}_{-\infty}$$

$t^2-25 \to 0^+$

$= \boxed{\text{Divergent}}$

Comparison test for improper integrals: Comparison Test Theorem. (CTT).

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $a \geq 0$

$\int f(x)dx \geq \int g(x)dx$

$5 \qquad \infty$

a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

$\int(\text{bigger})$ is convergent $\xrightarrow{\text{then}}$ $\int(\text{smaller})$ is also convergent

$\infty \qquad \boxed{\#}$

b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

$\int(\text{smaller})$ is divergent $\xrightarrow{\text{then}}$ $\int(\text{bigger})$ is also divergent

Ex: Test for convergence / divergence:

a) $\int_1^\infty e^{-x^2}dx$

for $1 \leq x < \infty.$

$1 \leq x^2 < \infty$

$x < x^2$

$e^x < e^{x^2}$

$e^{-x} > e^{-x^2}$

$\int_1^\infty e^{-x}dx > \int_1^\infty e^{-x^2}dx$

$\lim_{t\to\infty} \frac{e^{-x}}{-1}\Big|_1^t = -\lim_{t\to\infty}(e^{-t} - e^{-1}) = \frac{1}{e} \leftarrow$ convergent

By C.T.T. $\Rightarrow \int_1^\infty e^{-x^2}dx$ is also convergent

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$\frac{a}{b} \leq \frac{A}{b}$

$\frac{5}{27} \leq \frac{50}{2}$

$\frac{5}{57} \quad \frac{1}{12}$

Note: For convergence $\Rightarrow$ Construct bigger function $\frac{A}{b}$

For divergence $\Rightarrow$ Construct smaller function $\frac{a}{B}$

$$\int_1^\infty \frac{7x^2+5x+2}{4x^7+3x^2+1}\,dx \le \int_1^\infty \frac{\overbrace{7x^2+5x^2+2x^2}}{x^7}\,dx = \int_1^\infty \frac{14x^2}{x^7}\,dx = 14\int_1^\infty \frac{1}{x^5}\,dx$$

let $f(x)$

$p = 5 > 1$ is convergent by p-Test

Dominant terms: $\dfrac{x^2}{x^7} = \dfrac{1}{x^5} \Rightarrow p = 5 > 1$ is convergent

$\therefore$ by C.T.T. $\displaystyle\int_1^\infty f(x)\,dx$ is also convergent.

c) $$\int_2^\infty \frac{4x^3+3x+2}{\sqrt{\cancel{7}x^9+\cancel{x^8}+\cancel{3}}}\,dx \le \int_2^\infty \frac{4x^3+3x^3+2x^3}{\sqrt{x^9}}\,dx = \int_2^\infty \frac{9x^3}{\sqrt{x^9}}\,dx = 9\int_2^\infty \frac{x^3}{x^{9/2}}\,dx$$

Dominant terms: $\dfrac{x^3}{\sqrt{x^9}} = \dfrac{x^3}{x^{9/2}} = \dfrac{1}{x^{\frac{9}{2}-3}} = \dfrac{1}{x^{3/2}} > 1 \to$ Convergent

$$= 9\int_2^\infty \frac{1}{x^{\frac{9}{2}-3}}\,dx = 9\int_2^\infty \frac{1}{x^{3/2}}\,dx$$

$\Rightarrow p = \frac{3}{2} > 1 \Rightarrow$ Convergent by p-Test.

By C.T.T. $\Rightarrow \displaystyle\int_2^\infty f(x)\,dx$ is also convergent

d) $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$

$$\int_5^{\infty} \frac{3x^2+7x+4}{\sqrt[3]{5x^8+7x^5+7\cdot 1}}\,dx \geq \int_5 \frac{x^2}{\sqrt[3]{5x^8+7x^8+7x^8}}\,dx$$

Dominant terms: $\frac{x^2}{\sqrt[3]{x^8}} = \frac{x^2}{x^{8/3}} = \frac{1}{x^{8/3-2}} = \frac{1}{x^{2/3}}$, $2/3<1 \Rightarrow$ divergence.

$$= \int \frac{x^2}{\sqrt[3]{19x^8}}\,dx = \frac{1}{\sqrt[3]{19}} \int_5^{\infty} \frac{x^2}{x^{8/3}}\,dx = \frac{1}{\sqrt[3]{19}} \int_5^{\infty} \frac{1}{x^{\frac{8}{3}-2}}\,dx$$

$$= \frac{1}{\sqrt[3]{19}} \int_5^{\infty} \frac{1}{x^{2/3}}\,dx \Rightarrow p = \frac{2}{3} < 1 \Rightarrow \text{divergent by p-Test}$$

$\therefore$ By C.T.T. $\Rightarrow \int_5^{\infty} f(x)\,dx$ is also divergent.

e) $\int_1^{\infty} \frac{\sin(5x-3)+4\tan^{-1}(3x)}{x^3+\cancel{3x+4}}\,dx \leq \int_1^{\infty} \frac{\pi+7\pi}{x^3}\,dx = 8\pi \int_1^{\infty} \frac{1}{x^3}\,dx$ $\left\{ p=3>1 \text{ is convergent by p-Test.} \right.$

Construct a bigger funct.

Dominant terms: $\frac{1+4\cdot\frac{\pi}{2}}{x^3+1} \Rightarrow$ convergent $\Rightarrow$ Construct a bigger funct.

$\therefore$ by C.T.T. $\Rightarrow \int_1^{\infty} f(x)\,dx$ is also convergent.

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ex: $\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx \geq \int_1^{\infty} \frac{1}{x}\,dx$ $\left\{ p=1\leq 1 \Rightarrow \text{divergent by p-Test} \right.$

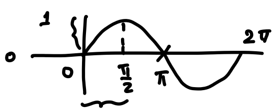
Dominant terms: $\frac{1}{x} \leftarrow$ divergent

$\therefore$ by C.T.T $\Rightarrow \int_1^{\infty} f(x)\,dx$ is also divergent.

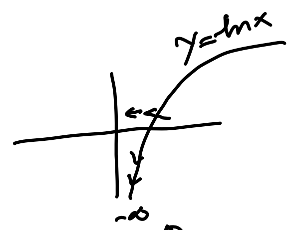
f) $\int_1^\infty e^{x^2+x+1}\,dx \geq \int_1^\infty e^1\,dx = e\int_1^\infty 1\,dx = e\int_1^\infty \frac{1}{x^0}\,dx \Rightarrow p=0<1$

is divergent by p-Test.

$\therefore$ by C.T.T. $\int_1^\infty f(x)\,dx$ is also divergent



g) $\int_0^{\pi/2} \frac{1}{x\sin(x)}\,dx$



We know $0 \leq \sin x \leq 1$

multiply by x: $0 \leq x\sin x \leq x$.

$\int_{a>0}^{\infty}$

$\frac{1}{x\sin x} \geq \frac{1}{x}$

$5 < 12$

$\frac{1}{5} > \frac{1}{12}$

$\Rightarrow \int_0^{\pi/2} \frac{1}{x\sin x}\,dx \geq \int_0^{\frac{\pi}{2}} \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln|x|\Big|_t^{\pi/2} = \lim_{t\to 0^+}\left[\ln\frac{\pi}{2} - \ln|t|\right] = \infty$.

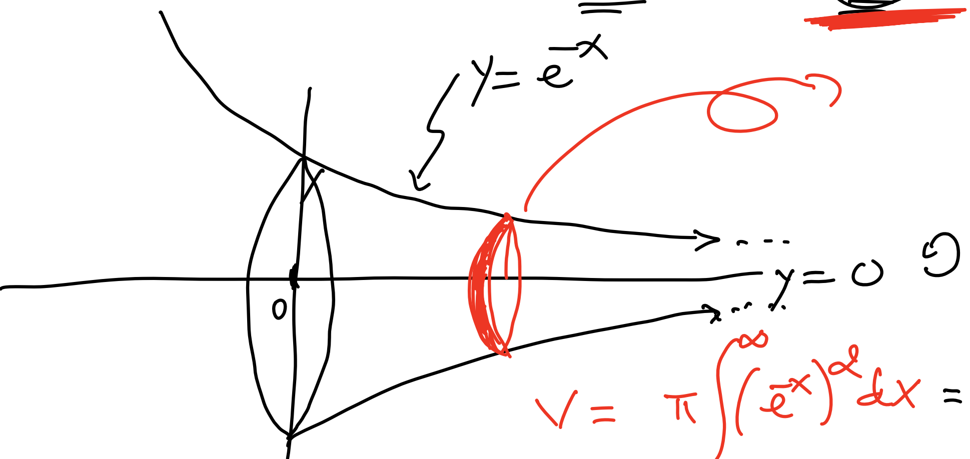
0 t $\frac{\pi}{2}$

$\therefore$ by C.T.T. $\int_0^{\frac{\pi}{2}} \frac{1}{x\sin x}\,dx$ is divergent.

$$\boxed{\int_0^{\pi} \frac{1}{x^5\cos^2 x}\,dx.}$$ ← Try this problem

Ex: Find the volume of the following:

a) The region bounded by $y = e^{-x}$; $y = 0$; for $x \geq 0$ is rotated about the x – axis.



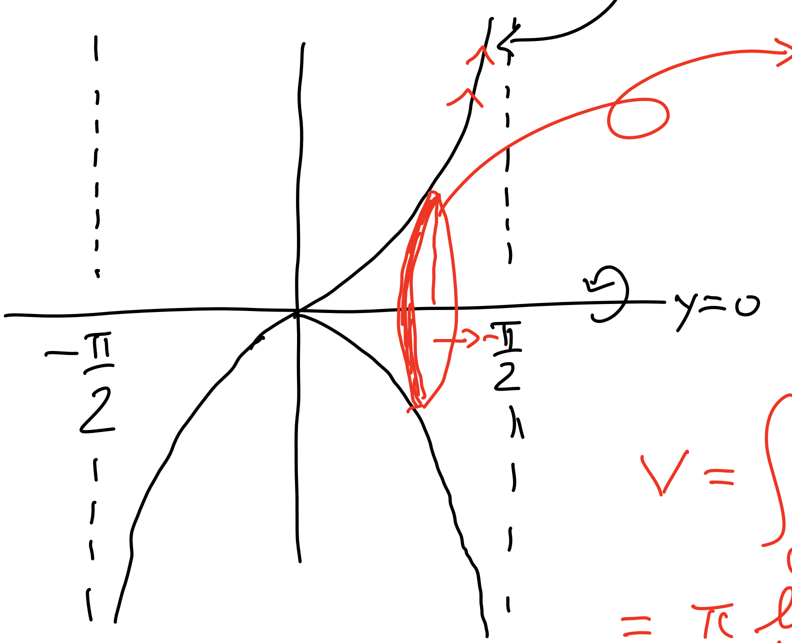
$V = \pi \cdot r^2 \cdot dx$, where r is $y_{top} = e^{-x}$, $y_{bot} = 0$; $r = e^{-x} - 0 = e^{-x}$.

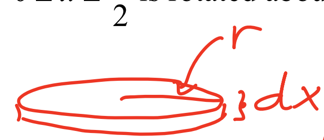
$$V = \pi \int_0^{\infty} \left(e^{-x}\right)^2 dx = \pi \lim_{t\to\infty} \int_0^t e^{-2x}\,dx = \pi \lim_{t\to\infty} \left.\frac{e^{-2x}}{-2}\right|_0^t$$

$$= -\frac{\pi}{2} \lim_{t\to\infty}\left(e^{-2t} - e^0\right) = -\frac{\pi}{2}\lim_{t\to\infty}\left(\frac{1}{e^{2t}} - 1\right)$$

$$= \boxed{\frac{\pi}{2}} \leftarrow \text{Convergent.}$$

b) The region bounded by $y = \tan(x)$; $y = 0$ for $0 \leq x \leq \frac{\pi}{2}$ is rotated about the x – axis.

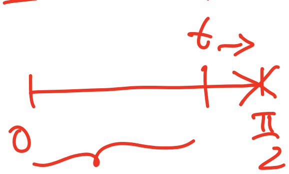




$V = \pi \cdot r^2 \cdot dx$. where r is $y_{top} = \tan x$, $y_{bot} = 0$

$r = \tan x - 0 = \tan x$.

$$V = \int_0^{\frac{\pi}{2}} \pi \left(\tan x\right)^2 dx$$



$$= \pi \lim_{t\to\frac{\pi}{2}^-} \int_0^t \tan^2 x\,dx = \pi \lim_{t\to\frac{\pi}{2}^-} \int_0^t \left(\sec^2 x - 1\right)dx$$

$$= \pi \lim_{t\to\frac{\pi}{2}^-} \left.\left(\tan x - x\right)\right|_0^t = \pi \lim_{t\to\frac{\pi}{2}^-}\left(\tan t - t\right) = \pi\left[\infty - \frac{\pi}{2}\right] = \infty$$

The volume is divergent