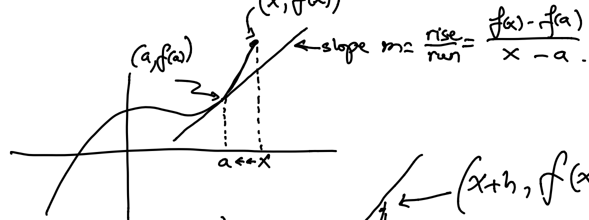
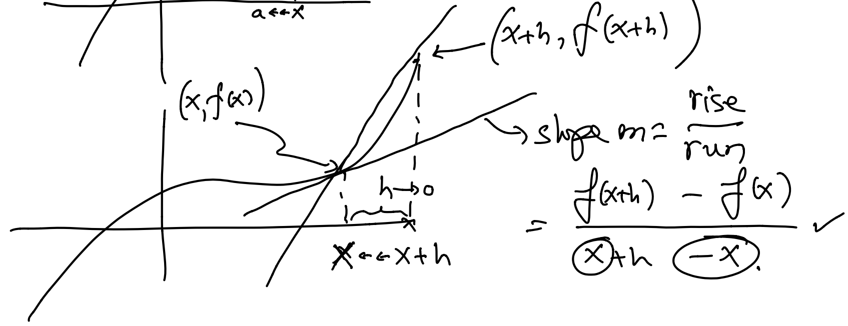
Review Math 180 materials:

Derivative:

1. Definition: Derivative of $f(x)$ at $x=a$.





$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivative at any x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2. Rules:

1. $(k)' = 0$
2. $(x^n)' = n x^{n-1}$
3. $(k \cdot f(x))' = k \cdot f'(x)$
4. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
5. $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + g'(x) \cdot f(x)$ $\quad [(uv)' = u'v + v'u]$
6. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$: $\left[\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}\right]$
7. $[(f \circ g)(x)]' = f'(g(x)) \cdot g'(x)$
8. (Exponential funct)' $\begin{cases} (e^x)' = e^x \\ (b^x)' = b^x \cdot \ln b. \\ (b^{u(x)})' = \underline{b^{u(x)}} \cdot \underline{u'(x)} \cdot \underline{\ln b}. \end{cases}$
9. (Log)' $\begin{cases} (\ln x)' = \frac{1}{x} \\ (\log_b x)' = \frac{1}{x} \cdot \frac{1}{\ln b}. \\ (\log_b(u(x)))' = \frac{1}{u(x)} \cdot u'(x) \cdot \frac{1}{\ln b}. \end{cases}$
10. (6 trig. functions)' $\begin{cases} (\sin x)' = \cos x & (\sec x)' = \sec x \tan x \\ (\cos x)' = -\sin x & (\csc x)' = -\csc x \cot x \\ (\tan x)' = \sec^2 x & (\cot x)' = -\csc^2 x. \end{cases}$
11. (Inverse Trig.)'

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$ $\quad$ $(\csc^{-1} x)' = \frac{-1}{x\sqrt{x^2-1}}$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$ $\quad$ $(\cot^{-1} x)' = \frac{-1}{1+x^2}$

3. Implicit Differentiation: ← Impossible to isolate the function y

Variable : $(x^m)' = m \cdot x^{m-1}$

function : $(y^n)' = n \cdot y^{n-1} \cdot y'$

Product Rule: $(x^m y^n)' = m x^{m-1} \cdot y^n + n \cdot y^{n-1} \cdot y' \cdot x^m.$

Ex: Differentiate the following functions:

a) $f(x)=\underbrace{3^{\sin(5x^2+1)}}_{u}\underbrace{\sqrt{\tan^{-1}(3x)+4x}}_{v}$ ✓

$$f'(x)=\underbrace{3^{\sin(5x^2+1)}\cdot\cos(5x^2+1)\cdot 10x\cdot\ln 3}_{u'}\cdot\underbrace{\sqrt{\tan^{-1}(3x)+4x}}_{v}+A$$

$$A=\underbrace{\frac{1}{2}\left(\tan^{-1}(3x)+4x\right)^{-\frac{1}{2}}\cdot\left[\frac{3}{1+9x^2}+4\right]}_{v'}\cdot\underbrace{3^{\sin(5x^2+1)}}_{u}.$$ ✓

$$y=f(x)$$

b) $f(x)=\cos^4\left(\dfrac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)$

$$f'(x)=\underbrace{4\cos^3\left(\frac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)}\cdot\underbrace{\left[-\sin\left(\frac{x^3-5x^2+1}{\sec^{-1}(3x+1)}\right)\right]}\cdot A$$

$$A=\frac{(3x^2-10x)\sec^{-1}(3x+1)-\dfrac{3}{(3x+1)\sqrt{(3x+1)^2-1}}(x^3-5x^2+1)}{\left[\sec^{-1}(3x+1)\right]^2}$$

Ex: Determine $\frac{dy}{dx} = y'$ for the expression: $\left(\sin^3\left(x^2+5y^3\right)\right)' + \left(e^{x^3y^2}\right)' = (4)'$

$$\underbrace{3\sin^2\left(x^2+5y^3\right)\cos\left(x^2+5y^3\right)}_{=A}\left(2x + 15y^2\cdot y'\right) + \underbrace{e^{x^3y^2}}_{=B}\left(3x^2y^2 + 2y\cdot y'x^3\right) = 0$$

$$y'\left(15y^2A + 2x^3yB\right) = -2xA - 3x^2y^2B.$$

$$y' = \frac{dy}{dx} = \frac{-2xA - 3x^2y^2B}{15y^2A + 2x^3yB}$$

2. Anti-derivative:- of $f(x)$ is another function $F(x)$, such that $F'(x) = f(x)$.

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{derivative}"] & f(x) \arrow[l, bend left, "\text{Anti-derivative}"] \arrow[r, "\text{derivative}"] & f'(x) \end{tikzcd}$$

1. $f(x) = K \Rightarrow F(x) = kx + C$
2. $f(x) = x^n \Rightarrow F(x) = \frac{x^{n+1}}{n+1} + C;\ \text{if } n \neq -1$
3. $f(x) = x^{-1} = \frac{1}{x} \Rightarrow F(x) = \ln|x| + C$
4. $f(x) = e^{kx} \Rightarrow F(x) = \frac{e^{kx}}{k} + C$
5. $f(x) = \sin(kx) \Rightarrow F(x) = -\frac{\cos(kx)}{k} + C$
6. $f(x) = \cos(kx) \Rightarrow F(x) = \frac{\sin(kx)}{k} + C$
7. $f(x) = \sec^2(kx) \Rightarrow F(x) = \frac{\tan(kx)}{k} + C$
8. $f(x) = \sec(kx)\tan(kx) \Rightarrow F(x) = \frac{\sec(kx)}{k} + C.$
9. $f(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow F(x) = \sin^{-1}(x) + C.$
10. $f(x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow F(x) = \sec^{-1}(x) + C.$
11. $f(x) = \frac{1}{1+x^2} \Rightarrow F(x) = \tan^{-1}x + C.$

3. Fundamental Theorem of Calculus (FTC)

FTC part I:
$$\begin{cases} f(x) = \int_a^x g(t)\,dt \Rightarrow f'(x) = g(x). \\ f(x) = \int_a^{u(x)} g(t)\,dt \Rightarrow f'(x) = g\left(u(x)\right)\cdot u'(x) \end{cases}$$

FTC part II:
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

(i.e. $F(x)$ is an anti-derivative of $f(x)$.)

Ex: Integrate the following: by method of substitution (u-sub method).

a) $\displaystyle\int \frac{7x-2}{\sqrt[3]{4x+3}}\,dx = \int \frac{7\left(\frac{u-3}{4}\right)-2}{\sqrt[3]{u}} \cdot \frac{du}{4}$

Let $u = 4x+3$

$du = 4\,dx$

$\frac{du}{4} = dx.$

$x = \frac{u-3}{4}$

$$= \frac{1}{4}\int \frac{\frac{7}{4}u - \frac{21}{4} - 2}{u^{1/3}}\,du.$$

$$= \frac{1}{4}\int\left(\frac{7}{4}u^{2/3} - \frac{29}{4}u^{-1/3}\right)du.$$

$$= \frac{1}{16}\left[7\cdot\frac{3}{5}\,u^{5/3} - 29\cdot\frac{3}{2}\,u^{2/3}\right] + C$$

$$= \frac{1}{16}\left[\frac{21}{5}(4x+3)^{5/3} - \frac{87}{2}(4x+3)^{2/3}\right] + C.$$

$\displaystyle\int \frac{7x-2}{\sqrt[3]{4x+3}}\,dx$

OR:

Let $u = \sqrt[3]{4x+3} \Rightarrow u^3 = 4x+3 \Rightarrow x = \frac{u^3-3}{4}$

$3u^2\,du = 4\,dx$

$\frac{3}{4}u^2\,du = dx$

$$= \int \frac{7\left(\frac{u^3-3}{4}\right)-2}{u} \cdot \frac{3}{4}u^2\,du = \frac{3}{4}\int\left(\frac{7}{4}u^3 - \frac{21}{4} - 2\right)u\,du$$

$$= \frac{3}{4}\int\left(\frac{7}{4}u^4 - \frac{29}{4}u\right)du.$$

$$= \frac{3}{16}\left[\frac{7}{5}\left(\sqrt[3]{4x+3}\right)^5 - \frac{29}{2}\left(\sqrt[3]{4x+3}\right)^2\right] + C$$

b) $\displaystyle\int \left(2+\sqrt{x}\right)^{12} dx$

Let $u = 2+\sqrt{x}$

$(u-2)^2 = (\sqrt{x})^2$

$(u-2)^2 = x.$

$2(u-2)\,du = dx$

$$\Rightarrow \int u^{12}\cdot 2(u-2)\,du = 2\int\left(u^{13} - 2u^{12}\right)du.$$

$$= 2\left[\frac{1}{14}\left(2+\sqrt{x}\right)^{14} - \frac{2}{13}\left(2+\sqrt{x}\right)^{13}\right] + C.$$

c) $\displaystyle\int x^8\sqrt[4]{4x^3+1}\,dx = \int x^6\cdot\sqrt[4]{4x^3+1}\cdot x^2\,dx.$

Let $u = \sqrt[4]{4x^3+1}$

$u^4 = 4x^3+1$

$4u^3\,du = 12x^2\,dx.$

$\frac{1}{3}u^3\,du = x^2\,dx.$

$x^3 = \frac{1}{4}(u^4-1)$

$x^6 = \frac{1}{16}(u^4-1)^2$

$$= \int \frac{1}{16}(u^4-1)^2\cdot u\cdot\frac{1}{3}u^3\,du. = \frac{1}{48}\int\left(u^8 - 2u^4 + 1\right)\cdot u^4\,du.$$

$$= \frac{1}{48}\int\left(u^{12} - 2u^8 + u^4\right)du.$$

$$= \frac{1}{48}\left[\frac{1}{13}\left(\sqrt[4]{4x^3+1}\right)^{13} - \frac{2}{9}\left(\sqrt[4]{4x^3+1}\right)^9 + \frac{1}{5}\left(\sqrt[4]{4x^3+1}\right)^5\right] + C.$$

d) $\displaystyle\int_{e/5}^{e^2/5} \frac{1\quad dx}{x\left[\ln(5x)+4\right]^7}$

Let $u = \ln(5x)+4$

$du = \frac{1}{\cancel{5}x}\cdot\cancel{5}dx = \frac{1}{x}dx$

$\begin{cases} x=\frac{e^2}{5} \Rightarrow u = \ln e^2+4 = 6. \\ x=\frac{e}{5} \Rightarrow u = \ln e+4 = 5 \end{cases}$

$$= \int_5^6 \frac{du}{u^7} = \int_5^6 u^{-7}du = \left.\frac{u^{-6}}{-6}\right|_5^6 = \frac{-1}{6}\left[\frac{1}{6^6} - \frac{1}{5^6}\right] = \dots = \#.$$

Nested Root.

e) $\int \sqrt{2+\sqrt{3x+1}}\,dx \rightsquigarrow$ Let $u = \sqrt{2+\sqrt{3x+1}}$

$u^2 = 2+\sqrt{3x+1} \Rightarrow (u^2-2)^2 = \left(\sqrt{3x+1}\right)^2$

$u^4 - 4u^2 + 4 = 3x+1$

$(4u^3 - 8u)du = 3dx.$

$\frac{4}{3}(u^3-2u)du = dx$

$\int u\cdot\frac{4}{3}(u^3-2u)du.$

$= \frac{4}{3}\int(u^4-2u^2)du = \frac{4}{3}\left[\frac{1}{5}u^5 - \frac{2}{3}u^3\right]+C.$

$= \frac{4}{3}\left[\frac{1}{5}\left(\sqrt{2+\sqrt{3x+1}}\right)^5 - \frac{2}{3}\left(\sqrt{2+\sqrt{3x+1}}\right)^3\right]+C.$

f) $\displaystyle\int_0^{1/3} \frac{\sqrt[5]{\tan^{-1}(3x)}}{1+9x^2}dx$

let $u = \sqrt[5]{\tan^{-1}(3x)}$ $\quad\begin{cases} x=\frac{1}{3} \Rightarrow u = \sqrt[5]{\frac{\pi}{4}} \\ x=0 \Rightarrow u=0 \end{cases}$

$u^5 = \tan^{-1}(3x)$

$5u^4du = \frac{1}{1+9x^2}\cdot 3\cdot dx.$

$\frac{5}{3}u^4du = \frac{1}{1+9x^2}dx.$

$$= \int_0^{\sqrt[5]{\frac{\pi}{4}}} u\cdot\frac{5}{3}u^4du = \frac{5}{3}\int_0^{\sqrt[5]{\frac{\pi}{4}}} u^5du.$$

$$= \frac{5}{3}\cdot\frac{1}{6}\cdot\left.u^6\right|_0^{\sqrt[5]{\frac{\pi}{4}}} = \frac{5}{18}\left(\sqrt[5]{\frac{\pi}{4}}\right)^6 = \dots = \#.$$

Section 7.1 **Integration by Parts**

$$\int (u\cdot v)' = \int u'v + \int v'u .$$

$$u\cdot v = \int v\,du + \boxed{\int u\cdot dv}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Tablet:

Derivative	u	dv	Anti-derivative
	du	v	

$= uv - \int v\,du$

$- \int$

Case1: Product of two different types of functions:

- polynomial and exponential functions
- polynomial and sine / cosine
- polynomial and log
- exponential and sine / cosine

Short Cut .

Case 2: One function

- log
- *Inverse trig functions*

Case 3: $\boxed{\text{Reduction formulas:}}$ → Reduce Power .

Ex: Integrate the following:

a) $\int \underbrace{(3x-5)}\underbrace{e^{2x}}dx$

$3x-5$	$e^{2x}dx$
$3dx$	$\frac{1}{2}e^{2x}$

$-\int$

$$= \underbrace{\frac{1}{2}(3x-5)e^{2x}}_{=A} - \frac{3}{2}\int e^{2x}dx$$

$$= A - \frac{3}{4}e^{2x} + C.$$

b) $\int \left(7x^3-5x^2+3\right)e^{3x+2}dx = e^{3x+2}\left[\frac{1}{3}(7x^3-5x^2+3) - \frac{1}{9}(21x^2-10x) + \frac{1}{27}(42x-10) - \frac{42}{81}\right] + C$

Take derivative to Zero

$7x^3-5x^2+3$		$e^{3x+2}dx$
$21x^2-10x$	$(+)$	$\frac{1}{3}e^{3x+2}$
$42x-10$	$(-)$	$\frac{1}{9}e^{3x+2}$
42	$(+)$	$\frac{1}{27}e^{3x+2}$
0	$(-)$	$\frac{1}{81}e^{3x+2}$

Anti-Derivative.

c) $\int \left(2x^2-5x+4\right)\sin(3x)dx = -\frac{1}{3}\cos(3x)(2x^2-5x+4) + \frac{1}{9}\sin(3x)(4x-5) + \frac{1}{27}\cos(3x)\cdot 4 + C.$

derivative to zero

$2x^2-5x+4$		$\sin(3x)dx$
$4x-5$	$(+)$	$-\frac{1}{3}\cos(3x)$
4	$(-)$	$-\frac{1}{9}\sin(3x)$
0	$(+)$	$\frac{1}{27}\cos(3x)$

Anti-Derivative.

d) $\int (4x^3-5x+2)\ln(2x)\,dx = \underbrace{\ln(2x)\left(x^4-\frac{5}{2}x^2+2x\right)}_{A} - \int\left(x^3-\frac{5}{2}x+2\right)dx.$

$\ln(2x)$	$(4x^3-5x+2)\,dx$
$\frac{1}{2x}\cdot 2dx = \frac{1}{x}dx$	$x^4-\frac{5}{2}x^2+2x$

$-\int$

$$= A - \left(\frac{1}{4}x^4 - \frac{5}{4}x^2 + 2x\right) + C.$$

e) $\int e^{3x}\cos(4x)\,dx = \underbrace{\frac{1}{4}e^{3x}\sin(4x)}_{A} - \frac{3}{4}\int e^{3x}\sin(4x)\,dx.$

e^{3x}	$\cos(4x)\,dx$
$e^{3x}\cdot 3dx$	$\frac{1}{4}\sin(4x)$

$-\int$

$$= A - \frac{3}{4}\left[-\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{4}\int e^{3x}\cos(4x)\,dx\right]$$

e^{3x}	$\sin(4x)\,dx$
$3e^{3x}dx$	$-\frac{1}{4}\cos(4x)$

$-\int$

$$= A + \frac{3}{16}e^{3x}\cos(4x) - \frac{9}{16}\int e^{3x}\cos(4x)\,dx$$

$$\left(1+\frac{9}{16}\right)\int e^{3x}\cos(4x)\,dx = A + \frac{3}{16}e^{3x}\cos(4x)$$

Ans: $\int e^{3x}\cos(4x)\,dx = \frac{16}{25}\left[A + \frac{3}{16}e^{3x}\cos(4x)\right] + C.$

f) $\int e^{\sqrt[3]{x}}\,dx = \int e^u \cdot 3u^2\,du = 3\int u^2 e^u\,du.$

Let $u = \sqrt[3]{x}$.
$u^3 = x$
$3u^2\,du = dx$

u^2	(+)	$e^u\,du$
$2u$	(−)	e^u
2	(+)	e^u
0		e^u

$= 3e^u\left[u^2 - 2u + 2\right] + C.$

$= 3e^{\sqrt[3]{x}}\left[(\sqrt[3]{x})^2 - 2\sqrt[3]{x} + 2\right] + C.$

g) $\int \cos\left(\sqrt[3]{x}\right)dx = \int \cos(u)\cdot 3u^2\,du = 3\int u^2\cos(u)\,du$

Let $u = \sqrt[3]{x}$

$u^3 = x$

$3u^2\,du = dx$

u^2	$(+)$ $\cos(u)\,du$
$2u$	$(-)$ $\sin(u)$
2	$(+)$ $-\cos(u)$
0	$-\sin(u)$

$= 3\left[u^2\sin(u) + 2u\cos(u) - 2\sin(u)\right] + C$

$= 3\left[\left(\sqrt[3]{x}\right)^2\sin\left(\sqrt[3]{x}\right) + 2\sqrt[3]{x}\cos\left(\sqrt[3]{x}\right) + 2\sin\left(\sqrt[3]{x}\right)\right] + C.$

h) $\int \tan^{-1}(5x)\,dx = \underbrace{x\tan^{-1}(5x)}_{A} - \int \frac{5x}{1+25x^2}\,dx$

$\tan^{-1}(5x)$	dx
$\frac{5}{1+25x^2}dx$	x

$-\int$

Let $u = 1+25x^2$

$\frac{du}{10} = \frac{50x\,dx}{10}.$

$\frac{du}{10} = 5x\,dx.$

$= A - \frac{1}{10}\int \frac{du}{u}.$

$= A - \frac{1}{10}\ln\left(1+25x^2\right) + C$

g) $\int \ln(3x+2)\,dx = \underbrace{x\ln(3x+2)}_{A} - \int \frac{3x+2-2}{3x+2}\,dx,$ $=1$

$\ln(3x+2)$	$dx.$
$\frac{3}{3x+2}dx$	x

$\int$

$= A - \int\left(1 - \frac{2}{3x+2}\right)dx.$

$= A - \left[x - \frac{2}{3}\ln|3x+2|\right] + C.$

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C}$$

Reduction formulas: $\int \sec^n x$, $\int \cos^n x$, $\int (\ldots)$

Secant: $(\tan x)' = \int \sec^2 x\,dx$

Ex: a) Prove the statement: $\int \sec^n x\,dx = \boxed{\dfrac{\sec^{n-2}x\tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad for\ n \neq 1}$

b) Use part (a) to evaluate $\int \sec^5(4x)\,dx$

Sol^n for (a): $\int \sec^n x\,dx = \int \sec^{n-2}x \cdot \boxed{\sec^2 x\,dx}$

Derivative	Antiderivative
$\sec^{n-2}x$	$\sec^2 x\,dx$
$(n-2)\sec^{n-3}x \cdot \sec x \cdot \tan x \cdot dx$	$\tan x$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\Rightarrow \int \sec^n x\,dx = \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x \cdot \tan^2 x\,dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x\,(\sec^2 x - 1)\,dx.$$

$$= \sec^{n-2}x\tan x - (n-2)\int (\sec^n x - \sec^{n-2}x)\,dx.$$

$$\int \sec^n x\,dx = \boxed{\sec^{n-2}x\tan x} - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx.$$

$$\underbrace{(1 + n-2)}_{= n-1}\int \sec^n x\,dx = \sec^{n-2}x\tan x + (n-2)\int \sec^{n-2}x\,dx.$$

$$\boxed{\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{(n-2)}{n-1}\int \sec^{n-2}x\,dx} \qquad n \neq 1$$

b) Use part (a) to evaluate: $\int \sec^6(4x)\,dx$

let $u = 4x$
$du = 4dx$
$\frac{du}{4} = dx.$

$$= \int \sec^6(u) \cdot \frac{du}{4} = \frac{1}{4}\int \sec^6(u)\,du = \frac{1}{4}\left[\frac{\sec^4 u\tan u}{5} + \frac{4}{5}\int \sec^4 u\,du\right]$$

$$= \frac{1}{20}\left[\sec^4 u\tan u + 4\left[\frac{\sec^2 u\tan u}{3} + \frac{2}{3}\int \sec^2 u\,du\right]\right]$$

$$= \frac{1}{20}\left\{ \sec^4(4x)\tan(4x) + \frac{4}{3}\sec^2(4x)\tan(4x) + \frac{8}{3}\tan(4x) \right] + C$$