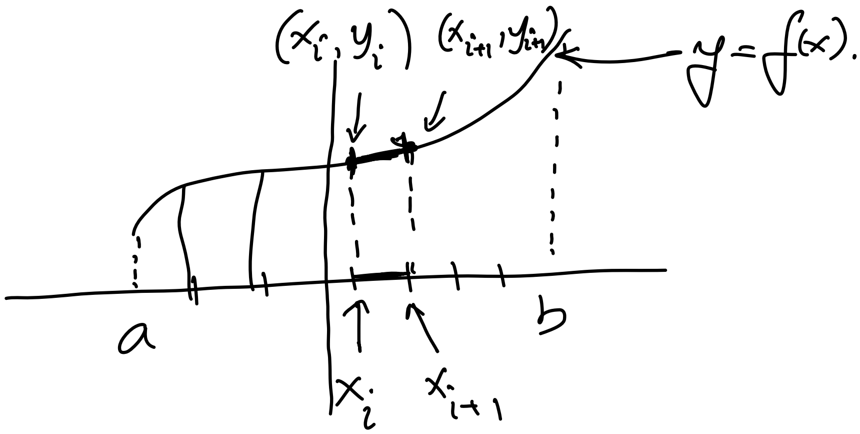
(x_1, y_1) (x_2, y_2) $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Section 8.1 $\boxed{\textit{Arc Length}}$

Given a function f(x) is continuous over interval [a,b]. How to find arc – length of f(x) over [a,b]



Distance between (x_i, y_i) & (x_{i+1}, y_{i+1})

$$d_i = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

Note : $x_{i+1} - x_i = \Delta x$.

$$\text{Length} = \sum d_i = \sum_{i=1}^{n} \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (y_{i+1}-y_i)^2}$$

such that

Apply the MVT over $[x_i, x_{i+1}] \Rightarrow \exists\, x_i^* \in (x_i, x_{i+1})$

$$y_{i+1} - y_i = f'(x_i^*)(x_{i+1}-x_i) = f'(x_i^*)\cdot\Delta x.$$

$$\text{Length}: L = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f'(x_i^*)\cdot\Delta x)^2} = \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\cdot\Delta x$$

$$\text{Exact length}: L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\cdot\Delta x = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$

Def: If f' is continuous on [a, b], then the length of the curve $y = f(x)$ for $a \le x \le b$, is $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

Or $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ If the function is in term of x Or $L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ If the function is in term of y.

General form:

Notation: $L = \int ds$ where $ds = \begin{cases} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(y')^2}\,dx & \text{if } y = f(x) \\ \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1+(x')^2}\,dy & \text{if } x = f(y) \end{cases}$

Ex: Find the arc – length of the following:

L

a) $y = x^{2/3}$ from (1, 1) to (8, 4)

$y = f(x)$

$$y' = \frac{2}{3}x^{-1/3} \Rightarrow (y')^2 = \left(\frac{2}{3}x^{-1/3}\right)^2 = \frac{4}{9}x^{-2/3}$$

$$L = \int ds = \int_1^8 \sqrt{1 + (y')^2}\,dx$$

$$= \int_1^8 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx = \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\,dx$$

$$= \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\,dx$$

let $u = \sqrt{9x^{2/3} + 4}$
$u^2 = 9x^{2/3} + 4$
$2u\,du = 6x^{-1/3}dx \Rightarrow \frac{1}{3}u\,du = \frac{1}{x^{1/3}}dx$

$x = 8 \Rightarrow u = \sqrt{40}$
$x = 1 \Rightarrow u = \sqrt{13}$

$$= \frac{1}{3}\int_{\sqrt{13}}^{\sqrt{40}} u \cdot \frac{1}{3}u\,du = \frac{1}{9}\int_{\sqrt{13}}^{\sqrt{40}} u^2\,du = \frac{1}{9}\cdot\frac{u^3}{3}\Big|_{\sqrt{13}}^{\sqrt{40}} = \frac{1}{27}\left[(\sqrt{40})^3 - (\sqrt{13})^3\right]$$

$= \ldots = \#.$

b) $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from x = 1 to x = 2.

$$L = \int ds = \int \sqrt{1 + (y')^2}\,dx$$

$$y = \frac{x^2}{4} - \frac{\ln x}{2} \Rightarrow (y')^2 = \left(\frac{1}{2}x - \frac{1}{2x}\right)^2$$

$$(y')^2 = \underset{a^2}{\frac{1}{4}x^2} - \underset{2ab}{\cancel{2}\cdot\frac{1}{\cancel{2}}\cancel{x}\cdot\frac{1}{2\cancel{x}}} + \underset{b^2}{\frac{1}{4x^2}}$$

$$(y')^2 = \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x - \frac{1}{2x}\right)^2$$

$$1 + (y')^2 = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$$

$$L = \int_1^2 \cancel{\sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^{\cancel{2}}}}\,dx = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right)dx.$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\ln|x|\,\Big|_1^2 = 1 + \frac{1}{2}\ln 2 - \frac{1}{4} = \boxed{\frac{3}{4} + \frac{1}{2}\ln 2}$$

c) $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right)$ *from* $x = 1$ *to* $x = 2$ ✓

$y = f(x)$

d) $y = \dfrac{x^3}{3} + x^2 + x + \dfrac{1}{4x+4}$ *for* $0 \le x \le 2$

$$L = \int ds = \int \sqrt{1 + (y')^2}\,dx \text{ where}$$

$$y = \tfrac{1}{3}x^3 + x^2 + x + (4x+4)^{-1} \Rightarrow y' = x^2 + 2x + 1 - (4x+4)^{-2} \cdot 4 = (x+1)^2 - \frac{4}{16(x+1)^2}$$

$$(y')^2 = \left[(x+1)^2 - \frac{1}{4(x+1)^2}\right]^2 \checkmark$$

$$(y')^2 = \overset{a^2}{(x+1)^4} \overset{-2ab}{- \underbrace{2 \cdot \tfrac{1}{4}}_{\frac{1}{2}}} \overset{+\ b^2}{+ \frac{1}{16(x+1)^4}}$$

$$1 + (y')^2 = (x+1)^4 + \frac{1}{2} + \frac{1}{16(x+1)^4} = \left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2$$

$$L = \int_0^2 \sqrt{\left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2}\,dx.$$

$$= \int_0^2 \left[(x+1)^2 + \frac{1}{4(x+1)^2}\right] dx$$

let $u = x + 1 \rightarrow \begin{cases} x = 2 \Rightarrow u = 3 \\ x = 0 \Rightarrow u = 1 \end{cases}$

$du = dx$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du = \frac{1}{3}u^3 - \frac{1}{4}u^{-1}\Big]_1^3 = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = \#.$$

$\checkmark\ x = f(y)$

$x = \frac{y^{3/2}}{3} - y^{1/2}$ from y = 2 to y = 9.

$L = \int ds = \int \sqrt{1 + (x')^2}\, dy$ where

$$\begin{cases} x = \frac{1}{3} y^{3/2} - y^{1/2} \Rightarrow x' = \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2} \\ (x')^2 = \left(\frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}\right)^2 = \frac{1}{4} y - \frac{1}{2} + \frac{1}{4} y^{-1} \\ 1 + (x')^2 = \frac{1}{4} y + \frac{1}{2} + \frac{1}{4} y^{-1} = \left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2 \end{cases}$$

$$L = \int_2^9 \sqrt{\left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2}\, dy = \int_2^9 \left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right) dy.$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \cdot y^{3/2} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} \cdot y^{1/2} \Big|_2^9 = \frac{1}{3}(27) + 3 - \frac{1}{3} 2^{3/2} - 2^{1/2}.$$

$$= 12 - \frac{4}{3}\sqrt{2} - \sqrt{2}.$$

$$= \boxed{12 - \frac{7}{3}\sqrt{2}}$$

$x = f(y)$

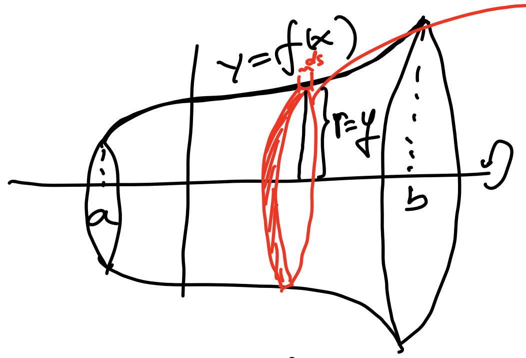
f) $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt \quad for \; -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

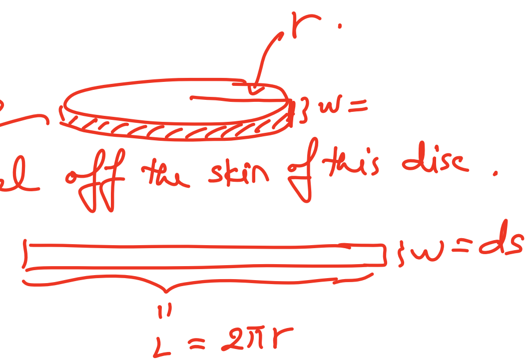
$L = \int ds = \int_{-\frac{\pi}{4}}^{\pi/4} \sqrt{1 + (x')^2}\, dy$ where $x' = \sqrt{\sec^4 y - 1} \Rightarrow (x')^2 = \sec^4 y - 1 \Rightarrow 1 + (x')^2 = \sec^4 y.$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^4 y}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y\, dy$$

$$= \tan y \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)$$

$$= 1 - (-1) = \boxed{2}$$





Surface-Area = $L \cdot W = 2\pi r \cdot ds$.

Surface-Area ?!

where ds = arc-length.

$$\boxed{\text{Surface area : } S = \int 2\pi r\, ds \text{ where } \begin{cases} ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y=f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x=f(y) \end{cases} \\ r = \begin{cases} y & \text{if rotated about the } x\text{-axis} \\ x & \text{if } \text{——— } y\text{-axis.} \end{cases} \end{cases}}$$

$r = y$ $\quad y = f(x)$

Ex: Find the area of the surface formed by rotating $\boxed{\text{about the x-axis}}$ the arc $y = \dfrac{x^3}{3}$ from x = 0 to x = 2.

$$S = 2\pi \int r\,ds \text{ where } \begin{cases} ds = \sqrt{1+(y')^2}\,dx \\ r = , \quad y^2 = \dfrac{x^3}{3} \end{cases}$$

$$y = \frac{x^3}{3} \Rightarrow y' = x^2 \Rightarrow (y')^2 = x^4 \Rightarrow 1+(y')^2 = 1+x^4.$$

$$S = 2\pi \int_0^2 \frac{x^3}{3}\sqrt{1+x^4}\,dx = \frac{2\pi}{3}\int_0^2 x^3\sqrt{1+x^4}\,dx$$

Let $u = \sqrt{1+x^4}$ $\quad \begin{cases} x=2 \Rightarrow u = \sqrt{17}. \\ x=0 \Rightarrow u = 1 \end{cases}$

$$\left.\begin{aligned} u^2 &= 1+x^4 \\ 2u\,du &= 4x^3\,dx \\ \tfrac{1}{2}u\,du &= x^3\,dx \end{aligned}\right\} = \frac{\cancel{2}\pi}{3}\int_1^{\sqrt{17}} u\cdot\frac{1}{\cancel{2}}u\,du = \frac{\pi}{3}\int_1^{\sqrt{17}} u^2\,du$$

$$= \frac{\pi}{3}\cdot\frac{1}{3}u^3\Big|_1^{\sqrt{17}} = \frac{\pi}{9}\left[(\sqrt{17})^3 - 1\right]$$

$= \#.$

$x = f(y)$

Ex. Find the surface area of the arc of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from (2/3, 1) to (14/3, 3).

a) Rotates about the y – axis. $\Rightarrow r = x$

$$S = 2\pi \int r\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{1 + (x')^2}\,dy \\ r = x = \frac{y^3}{6} + \frac{1}{2y} \end{cases} \quad \text{match}$$

$$x = \frac{y^3}{6} + \frac{1}{2}y^{-1} \Rightarrow (x')^2 = \left(\frac{1}{2}y^2 - \frac{1}{2}y^{-2}\right)^2 = \frac{1}{4}y^4 - \frac{1}{2} + \frac{1}{4}y^{-4}$$

$$1 + (x')^2 = \frac{1}{4}y^4 + \frac{1}{2} + \frac{1}{4}y^{-4} = \left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2$$

$$S = 2\pi \int_1^3 \left(\frac{y^3}{6} + \frac{1}{2y}\right)\sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2}\,dy = 2\pi \int_1^3 \left(\frac{y^3}{6} + \frac{1}{2y}\right)\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)dy$$

$$S = 2\pi \int_1^3 \left(\frac{1}{12}y^5 + \frac{1}{12}y + \frac{1}{4}y + \frac{1}{4}y^{-3}\right)dy = 2\pi \left[\frac{1}{72}y^6 + \frac{1}{6}y^2 - \frac{1}{8}y^{-2}\right]_1^3$$

$$= \dots = \#$$

b) Rotates about the x – axis. $\Rightarrow r = y$

$$S = 2\pi \int r\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{1 + (x')^2}\,dy \\ r = y \end{cases} \quad \text{already match}$$

$$1 + (x')^2 = \left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2$$

$$S = 2\pi \int_1^3 y\sqrt{\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)^2}\,dy = 2\pi \int_1^3 y\left(\frac{1}{2}y^2 + \frac{1}{2}y^{-2}\right)dy$$

$$= 2\pi \int_1^3 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-1}\right)dy = 2\pi \left[\frac{1}{8}y^4 + \frac{1}{2}\ln|y|\right]_1^3$$

$$= 2\pi \left[\frac{81}{8} + \frac{1}{2}\ln 3 - \frac{1}{8}\right] = \#.$$

$y = f(x)$ ✓ $\qquad r = y$

Ex: The curve $y = \sqrt{2x - x^2}$ for $0.5 \le x \le 1.5$ is rotated $\boxed{\text{about the x – axis.}}$ Find its surface area.

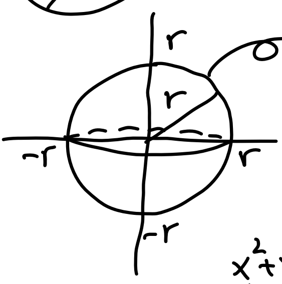
$$S = 2\pi \int r\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{1+(y')^2}\,dx & \text{match.} \\ r = y = \sqrt{2x - x^2} & ✓ \end{cases}$$

$$y = \sqrt{2x - x^2} \Rightarrow y' = \frac{1}{2}\left(2x - x^2\right)^{-1/2}(2-2x) = \frac{\cancel{2}(1-x)}{\cancel{2}\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$$

$$1+(y')^2 = \left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2 = \frac{(1-x)^2}{2x-x^2} + 1 = \frac{1-\cancel{2x}+\cancel{x^2}+\cancel{2x}-\cancel{x^2}}{2x-x^2} = \frac{1}{2x-x^2}$$

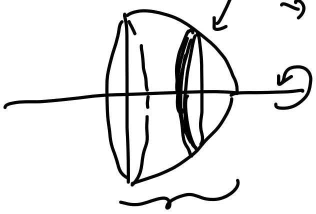
$$S = 2\pi \int_{0.5}^{1.5} \underbrace{\cancel{\sqrt{2x-x^2}}}_{r} \cdot \underbrace{\sqrt{\frac{1}{\cancel{2x-x^2}}}\,dx}_{ds} = 2\pi \int_{0.5}^{1.5} dx = 2\pi x \Big|_{0.5}^{1.5} = 2\pi(1.5-0.5) = \boxed{2\pi}$$

Ex: Show that the surface area of a sphere of radius r is $4\pi r^2$



$$x^2 + y^2 = r^2$$

$$S = 2\pi \int a\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{1+(y')^2}\,dx & \text{match.} \\ r = y = \sqrt{r^2 - x^2} \end{cases}$$



$$x^2 + y^2 = r^2 \rightarrow y = \pm\sqrt{r^2 - x^2}$$

where $y = \sqrt{r^2 - x^2} \Rightarrow y' = \frac{1}{\cancel{2}}\left(r^2 - x^2\right)^{-1/2} \cdot (-\cancel{2}x)$

$$1 + (y')^2 = \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2 = \frac{x^2}{r^2-x^2} + 1 = \frac{x^2 + r^2 - x^2}{r^2 - x^2}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$S = 2 \cdot 2\pi \int_0^r \cancel{\sqrt{r^2-x^2}} \sqrt{\frac{r^2}{\cancel{r^2-x^2}}}\,dx = 4\pi \int_0^r r\,dx = 4\pi r x \Big|_0^r = \boxed{4\pi r^2}$$

Arc-length: $L = \int ds$ where $ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y = f(x) \\ \text{or} \\ \sqrt{1+(x')^2}\,dy & \text{if } x = f(y) \end{cases}$ ✓

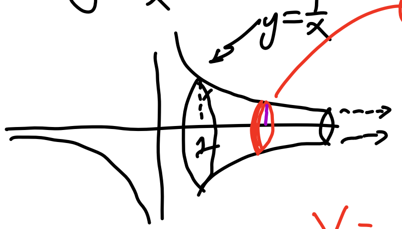
Surface-Area: $S = 2\pi \int r\,ds$ where $\begin{cases} ds = \ast \\ r = \begin{cases} y & \text{if rotated about x-axis} \\ x & \text{——— y-axis} \end{cases} \end{cases}$

Ex: The region bounded by $y = \frac{1}{x}$; $y = 0$ and $x \geq 1$ is rotated about the x – axis.

$y = f(x)$

a) Find its volume.

$y = \frac{1}{x}$



S / ds

dx

$V = \pi \cdot r^2 \cdot dx$

where r is $y_{top} = \frac{1}{x}$, $= \frac{1}{x} - 0 = \frac{1}{x}$.

$y_{bot} = 0$

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx = \pi \lim_{t \to \infty} \int_1^t x^{-2} dx. = \pi \lim_{t \to \infty} \frac{x^{-1}}{(-1)} \Big|_1^t.$$

$1 \longrightarrow t \to \infty$

$$= -\pi \lim_{t \to \infty} \left[\frac{1}{t} - 1\right] = \pi \leftarrow \text{Convergent}$$

$\sqrt{1 + (y')^2}\, dx$

$\sqrt{1 + (x')^2}\, dy$

b) Find it surface area.: $S = 2\pi \int r\, ds$

where $\begin{cases} ds = \sqrt{1 + (y')^2}\, dx \\ r = y = \frac{1}{x} \end{cases}$ match.

$y = \frac{1}{x} = x^{-1} \Rightarrow y' = -x^{-2} \Rightarrow (y')^2 = \left(-x^{-2}\right)^2 = x^{-4} = \frac{1}{x^4}$

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx \geq 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1}\, dx.$$

$$= 2\pi \int_1^{\infty} \frac{1}{x}\, dx$$

$p = 1 \geq 1$ is divergent by p – Test

$\Rightarrow S$ is divergent by C.T.T. $= \infty$

$V = \pi$.

Gabriel's horn