Section 9.3 ***Differential Equations*** (D.E.)

Def: A differential equation is any equation that involves one or more derivative of an unknown function. (i.e. solution(s) of DE are functions)

Where do differential equations come from?

Population:

Newton's Law of Cooling: :

Second Newton's Law: :

Ex: Solve:

a) $y' = y$ (1st) $\Rightarrow y(x) = Ae^x$

b) $y'' = -y$ (2nd) $\left\{ \begin{array}{l} y_1 = A\sin x \\ + \\ y_2 = B\cos x \end{array} \right\} y = y_1 + y_2 = A\sin x + B\cos x$

Def: A separable equation is a first – order differential equation in which the expression for $\dfrac{dy}{dx}$ can be factored as a function of x times a function of y. fraction $= \dfrac{dy}{dx} = f(x) \cdot g(y)$

$\underbrace{\text{function of x}}_{f(x)} \cdot \underbrace{\text{function of y}}_{g(y)}$

$$\int \frac{dy}{g(y)} = \int f(x) \cdot dx$$

Ex: Solve the following DE:

a) $\dfrac{dy}{dx} = \dfrac{e^{2x-3y}}{e^{2y-3x+2}} = f(x) \cdot g(y)$

$$\frac{dy}{dx} = e^{2x-3y-(2y-3x+2)} = e^{5x-5y-2}.$$

$$\boxed{\begin{array}{l} \rightarrow \dfrac{b^m}{b^n} = b^{m-n}. \\ \rightarrow b^m \cdot b^n = b^{m+n} \\ \rightarrow (b^m)^n = b^{m \cdot n} \end{array}}$$

$$\frac{dy}{dx} = \underbrace{e^{5x-2}}_{f(x)} \cdot \underbrace{e^{-5y}}_{g(y)}$$

$$\int \frac{dy}{e^{-5y}} = \int e^{5x-2}\,dx. \Rightarrow \underbrace{\int e^{5y}\,dy}_{} = \int e^{5x-2}\,dx$$

$$\frac{1}{5}e^{5y} = \frac{1}{5}e^{5x-2} + C$$

$$e^{5y} = e^{5x-2} + C$$

$$5y = \ln\left(e^{5x-2} + C\right)$$

$$\boxed{y = \frac{1}{5}\ln\left(e^{5x-2} + C\right)}$$

b) $\frac{dy}{dx} = 3x^2y^2 - 5xy^2 + 3y^2 = f(x) \cdot g(y)$

$$\frac{dy}{dx} = y^2 \cdot (3x^2 - 5x + 3)$$

$$\int \frac{dy}{y^2} = \int (3x^2 - 5x + 3)\,dx . \Rightarrow$$

$$\underbrace{\int y^{-2}\,dy}_{-y^{-1}} = \int (3x^2 - 5x + 3)\,dx .$$

$$-y^{-1} = x^3 - \frac{5}{2}x^2 + 3x + C$$

$$\frac{1}{y} = -x^3 + \frac{5}{2}x^2 - 3x + C$$

$$\Rightarrow \boxed{y = \frac{1}{C - x^3 + \frac{5}{2}x^2 - 3x}}$$

Ex: Initial Value Problem (IVP)

a) $\frac{dy}{dx} = \frac{2x+1}{2y}$; $y(-2) = -1$ $\begin{cases} x = -2 \\ y = -1 . \end{cases}$

$$\underbrace{\int 2y\,dy}_{y^2} = \int (2x+1)\,dx .$$

$$y^2 = x^2 + x + C$$

$$(-1)^2 = (-2)^2 - 2 + C .$$

$$1 = 4 - 2 + C$$

$$1 = 2 + C \Rightarrow C = -1 .$$

Sol: $y^2 = x^2 + x - 1 .$

$$\boxed{y = \pm\sqrt{x^2 + x - 1}}$$

$$\boxed{\frac{dy}{dx} = f(x) \cdot g(y) \qquad \int \frac{dy}{g(y)} = \int f(x)\,dx}$$

b) $\dfrac{dy}{dx} = 3y + 2;\ y(1) = 3$ $\left\{ \begin{array}{l} x = 1 \\ y = 3 \end{array} \right.$

$$\underbrace{\int \frac{dy}{3y+2}} = \underbrace{\int dx}$$

$$\frac{1}{3}\ln|3y+2| = x + C$$

$$\ln|3y+2| = 3x + C$$

$$3y + 2 = e^{3x+C} = e^{3x}\cdot\overbrace{\boxed{e^C}}^{=k} = ke^{3x}.$$

$$\frac{\cancel{3}y}{\cancel{3}} = \frac{k}{3}e^{3x} - \frac{2}{3}.$$

$$y = ke^{3x} - \frac{2}{3}$$

$$3 = ke^{3} - \frac{2}{3}$$

$$\underbrace{3 + \frac{2}{3}} = k\cdot e^{3}$$

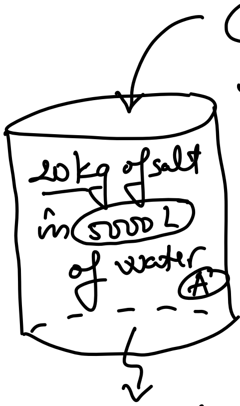
$$\frac{11}{3} = ke^{3} \Rightarrow k = \frac{11}{3e^{3}}$$

$$\text{Ans: } y = \frac{11}{3e^3}\cdot e^{3x} - \frac{2}{3} = \frac{1}{3}\left[11e^{3x-3} - 2\right]$$

Mixing Problems:

Ex: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 20L/min. ~~The solution~~ is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

$\underbrace{20 L/\text{min}}$ @ $\underbrace{0.03 \text{kg}/L}$. =



$20 L/\text{min}$.

sol: Let $A(t)$ be the amount of salt in the tank at time t (in mins.).

$$\frac{dA}{dt} = \underbrace{\text{rate in}} - \underbrace{\text{rate out}}.$$

$$= \underbrace{(20)(0.03)} - \frac{A}{500\cancel{0}}\cdot 2\cancel{0}$$

$$\boxed{\frac{dA}{dt} = \left(0.6 - \frac{1}{250}A\right)\cdot 1;\quad A(0) = 20}$$

$$\underbrace{\int \frac{dA}{0.6 - \frac{1}{250}A}} = \underbrace{\int dt}$$

$$-250\ln\left|0.6 - \frac{1}{250}A\right| = t + C$$

$$\ln\left|0.6 - \frac{1}{250}A\right| = -\frac{1}{250}t + C.$$

$$0.6 - \frac{1}{250}A = e^{-\frac{1}{250}t + C} = e^{-\frac{1}{250}t}\cdot\overbrace{\boxed{e^C}}^{=k} = ke^{-\frac{1}{250}t}$$

$$-250\left[0.6 - \frac{1}{250}A = ke^{-\frac{1}{250}t}\right]$$

$$-150 + A = ke^{-\frac{1}{250}t}$$

$$\boxed{A(t) = 150 + ke^{-\frac{1}{250}t}}$$

$$A(0) = 150 + ke^{0} = 20$$

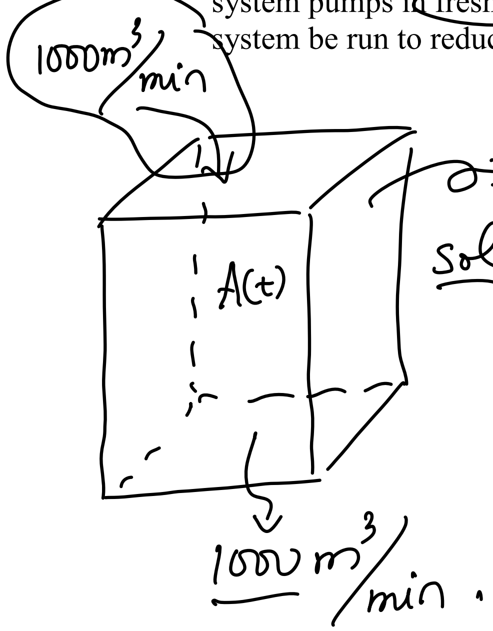
$$\boxed{A(t) = 150 - 130e^{-\frac{1}{250}t}}$$

$$k = -130$$

After 30 mins $\Rightarrow$ $A(30) = 150 - 130e^{-\frac{1}{250}(30)} = 34.7\text{ kg}$ ✓

fresh air

***Ex*:** A natural gas leak has filled a building enclosing 50,000 m^3 with a 1 percent mixture of natural gas and air. The gas line is shut off, and an emergency ventilation system pumps in fresh air at the rate of 1000 m^3/min. How long must the ventilation system be run to reduce the concentration of natural gas to 0.01 percent?



$\rightarrow$ Volume $= 50{,}000\text{ m}^3$

Sol: Let $A(t)$ be the amount of natural gas in the building at time t (in mins.)

$$\frac{dA}{dt} = \underbrace{\text{rate in}} - \underbrace{\text{rate out}}.$$

$$= (1000)\cdot 0 - \frac{A}{50{,}000}\cancel{1000}$$

$$\underbrace{\frac{dA}{dt} = -\frac{1}{50}A} \; ; \quad A(0) = 1\% \text{ of } 50{,}000 = (0.01)(50{,}000)$$

$$\boxed{A(0) = 500}$$

$$\underbrace{\int \frac{dA}{A} = -\frac{1}{50}\int dt}.$$

$$\ln|A| = -\frac{1}{50}t + C$$

$$A = e^{-\frac{1}{50}t + C} = e^{-\frac{1}{50}t}\cdot \boxed{e^C} = k$$

$$A(t) = ke^{-\frac{1}{50}t}.$$

$$A(0) = K\cdot e^0 = 500$$

$$K = 500.$$

$$A(t) = 500e^{-\frac{1}{50}t}$$

$t = ?$ such that $A(t) = 0.01\%$ of $50{,}000$.

$$= (0.0001)(50{,}000) = 5$$

$$A(t) = 5 \Rightarrow \frac{500e^{-\frac{1}{50}t}}{500} = \frac{5}{500}$$

$$e^{-\frac{1}{50}t} = 0.01 \Rightarrow \ln\left(e^{-\frac{1}{50}t}\right) = \ln(0.01)$$

$-\frac{1}{50}t = \ln(0.01) \Rightarrow t = -50\ln(0.01) = 230.\ldots$ mins.

≈ 4 hrs.

Ex: Mortgage: Determine the monthly payment of a loan of \$650,000 at interest rate of 4.5% per year compounded continuously for 30 years. Then determine the total interest of the loan when it's paid off after 30 years.

Interest / yr

balance $B(t)$

payment / yr

Sol: Let $B(t)$ be the balance of the mortgage at time t (in years).

Let M be the monthly payment.

$$\frac{dB}{dt} = \underbrace{\text{rate in}} - \underbrace{\text{rate out}}$$

$$= 0.045B - 12M.$$

$$\boxed{B(0) = 650{,}000}\ ; \quad B(30) = 0$$

$$\frac{dB}{dt} = (0.045B - 12M)\cdot 1.$$

$$\underbrace{\int \frac{dB}{0.045B - 12M}} = \underbrace{\int dt}.$$

$$\frac{1}{0.045}\ln|0.045B - 12M| = t + C.$$

$$\ln|0.045B - 12M| = 0.045t + C.$$

$$0.045B - 12M = e^{0.045t + C} = e^{0.045t}\cdot \underbrace{\boxed{e^C}}_{=K}.$$

$$0.045B - 12M = Ke^{0.045t}.$$

$$\frac{\cancel{0.045}B}{\cancel{0.045}} = \frac{K}{0.045}e^{0.045t} + \frac{12M}{0.045}.$$

$0.045 \quad 0.045$

$$B(t) = \overset{\shortparallel}{K} e^{0.045t} + 266.67M.$$

$$B(0) = K \cdot \boxed{e^{0}}^{=1} + 266.67M = 650,000.$$

$$K = 650,000 - 266.67M$$

$$B(t) = (650,000 - 266.67M)\, e^{0.045t} + 266.67M.$$

$$B(30) = (650,000 - 266.67M)\, \underbrace{e^{0.045(30)}}_{\approx 3.86} + 266.67M = 0$$

$$= 2,509,000 - \underbrace{1,029.35M + 266.67M}_{} = 0$$

$$2,509,000 - 762.68M = 0.$$

$$M = \frac{2,509,000}{762.68} = \underbrace{\$3,289.\tfrac{72}{xx}}_{\text{Monthly Payment}}.$$

Total Cost = Total Interest:

$$\underline{\left(3,289.\tfrac{72}{xx}\right)(12)(30)} - 650,000 = \$534,299.\tfrac{20}{xx}$$