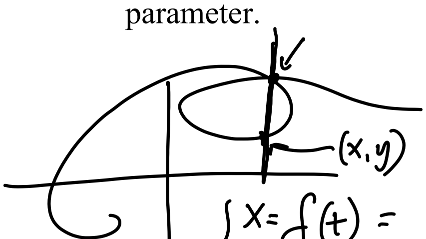
Chapter ~~10~~ 9 Parametric Equations and Polar Coordinates

Section ~~10~~9.1 Curves Defined by Parametric Equations

***Def*:** Let $x = f(t)$ and $y = g(t)$, where f and g are two functions whose common domain is some vertical I. the collection of points defined by $(x, y) = (f(t), g(t))$ is called a plane curve. The equations $x = f(t)$ and $y = g(t)$ where t is in I, are called parametric equations of the curve. The variable t is called a parameter.



$\begin{cases} x = f(t) = \\ y = g(t) \end{cases}$

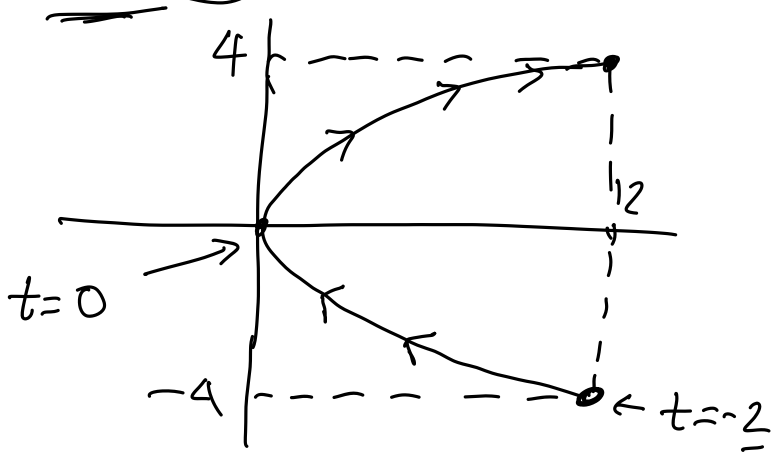
$$\boxed{\begin{cases} x = f(t) \\ y = g(t) \end{cases} \text{ for } t \in I.}$$

Ex: Sketch the graph of the following parametric equations and indication the direction on the graph.

a) $\begin{cases} x = 3t^2 = f(t) \\ y = 2t = g(t) \end{cases}$; *for* $\boxed{-2 \le t \le 2}$

Eliminate the parameter t.

$$\begin{cases} x = 3t^2 \\ y = 2t \Rightarrow t = \frac{1}{2}y \end{cases} \quad x = 3\left(\frac{1}{2}y\right)^2 \Rightarrow \boxed{x = \frac{3}{4}y^2}$$



$t = -2 \Rightarrow \begin{cases} x = 3t^2 \\ y = 2t \end{cases}\Big|_{t=-2} \Rightarrow \begin{cases} x = 3(-2)^2 = 12 \\ y = 2(-2) = -4 \end{cases}$

$t = 0 \begin{cases} x = 3(0)^2 = 0 \\ y = 2(0) = 0 \end{cases}$

$t = 2 \begin{cases} x = 3(2)^2 = 12 \\ y = 2(2) = 4 \end{cases}$

$\frac{t}{5}$

$\frac{2\pi}{5}$

$5t$

$$\begin{cases} x = 2\cos t + 2 \\ y = 3\sin t - 1 \end{cases} \quad t \in \mathbb{R}$$

$x = f(t)$, $y = g(t)$

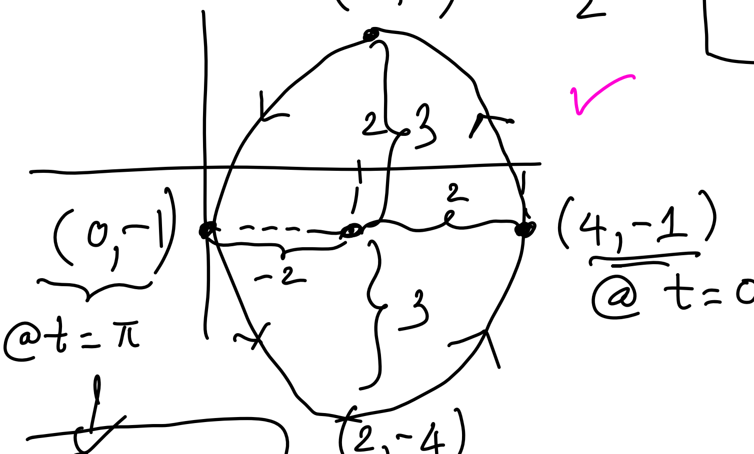
t

$$\begin{cases} \cos t = \dfrac{x-2}{2} \\ \sin t = \dfrac{y+1}{3} \end{cases} \xrightarrow[\text{both sides}]{\text{square}} + \begin{cases} \cos^2 t = \dfrac{(x-2)^2}{4} \\ \sin^2 t = \dfrac{(y+1)^2}{9} \end{cases}$$

$$\boxed{1 = \frac{(x-2)^2}{4} + \frac{(y+1)^2}{9}}$$

Elipse centered at $(2,-1)$

$(2,2)$ @ $t = \frac{\pi}{2}$



Direction:

$$t = 0 \Rightarrow \begin{cases} x = 2\cos(0) + 2 = 4 \\ y = 3\sin(0) - 1 = -1 \end{cases}$$

$$t = \frac{\pi}{2} \quad \begin{cases} x = 2\cos\frac{\pi}{2} + 2 = 2 \\ y = 3\sin\frac{\pi}{2} - 1 = 2. \end{cases}$$

$$t = \pi \quad \begin{cases} x = 2\cos\pi + 2 = 0 \\ y = 3\sin\pi - 1 = -1 \end{cases}$$

$$\begin{cases} x = 3\sin^2 t - 1 \\ y = 2\cos t + 3 \end{cases}; \ \cancel{-2 \le t \le 2} \quad t \in \mathbb{R}.$$

$$\sin^2 t = \frac{x+1}{3}$$
$$+ \qquad +$$
$$\cos^2 t = \left(\frac{y-3}{2}\right)^2$$

$$1 = \frac{x+1}{3} + \frac{(y-3)^2}{4}$$

$$3 = x + 1 + \frac{3}{4}(y-3)^2$$

$$x = 3 - 1 - \frac{3}{4}(y-3)^2$$

$$\boxed{x = -\frac{3}{4}(y-3)^2 + 2} \Rightarrow \text{vertex } (2,3)$$

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$x = 3\sin^2 t - 1$

$0 \le \sin^2 t \le 1$

$0 \le 3\sin^2 t \le 3$

$-1 \le \underbrace{3\sin^2 t - 1}_{=x} \le 2.$

$\boxed{-1 \le x \le 2}$

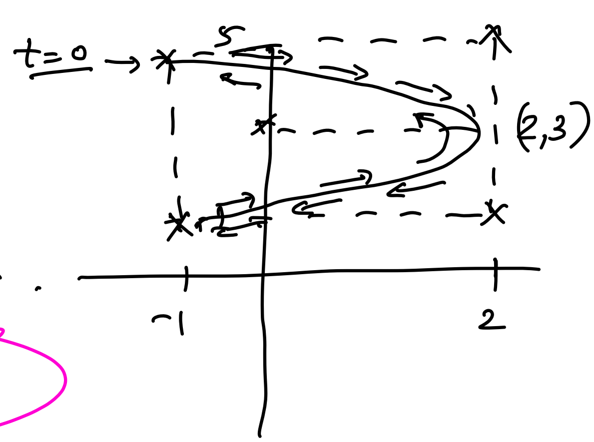
$y = 2\cos t + 3$

$-1 \le \cos t \le 1$

$-2 \le 2\cos t \le 2$

$1 \le 2\cos t + 3 \le 5.$

$\boxed{1 \le y \le 5}$



direction $t=0$ $\begin{cases} x = 3\sin^2(0) - 1 = -1 \\ y = 2\cos(0) + 3 = 5 \end{cases}$

d) $\begin{cases} x = (v_0 \cos\theta)t \\ y = -\dfrac{1}{2}gt^2 + (v_0 \sin\theta)t + h \end{cases}$ (Time as a Parameter: Projectile Motion)

gravity : $\begin{cases} 9.8 \text{ m/sec}^2 \\ \text{or} \\ 32 \text{ ft/sec}^2 \end{cases}$

$V_0 = 150$ ft/sec $\quad \theta = 30° = \frac{\pi}{6}$.

Ex: Suppose that Jim hit a golf ball with an initial velocity of 150 ft/sec. at an angle of 30 degree to the horizontal.

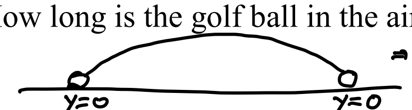
a) Find parametric equations that describe the position of the ball as a function of time.

$\begin{cases} x = f(t) = (V_0 \cos\theta)t = (150 \cdot \cos\frac{\pi}{6})t = (150 \cdot \frac{\sqrt{3}}{2})t = 75\sqrt{3}t. \\ y = g(t) = -\frac{1}{2}gt^2 + (V_0 \sin\theta)t + h = -\frac{1}{2}(32)t^2 + 150 \cdot \sin\frac{\pi}{6} \cdot t + 0. \\ \qquad = -16t^2 + 75t. \end{cases}$

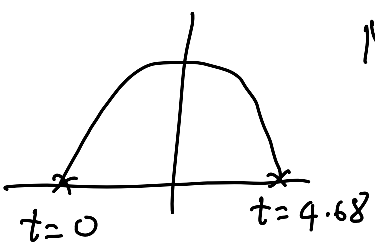
b) How long is the golf ball in the air?

$\Rightarrow y=0 \Rightarrow -16t^2 + 75t = 0 \Rightarrow t(-16t+75) = 0 \Rightarrow t=0,\ \frac{75}{16} = 4.68$ sec.

$0 \le t \le 4.68$ sec.



c) When is the ball at its maximum height? Determine the maximum height of the ball.

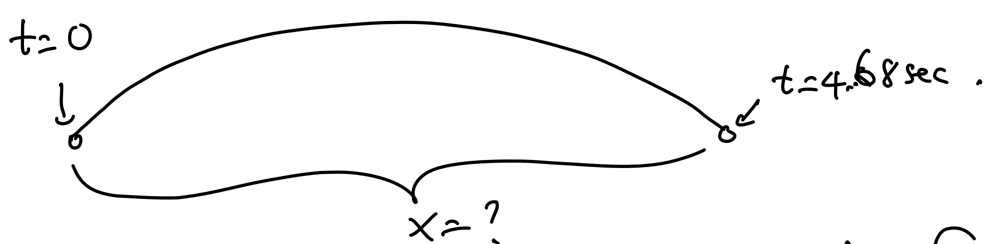


Max. height occurred @ $t = \frac{4.68}{2} = 2.34$ sec.

Max. height $\Rightarrow y = -16t^2 + 75t \Big|_{t=2.34} = -16(2.34)^2 + 75(2.34)$

$\Rightarrow y = 87.89$ ft.

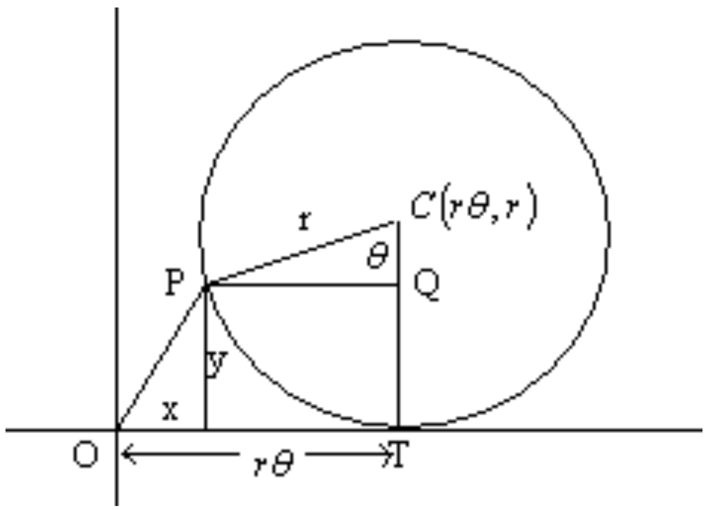
d) Determine the distance that the ball traveled.



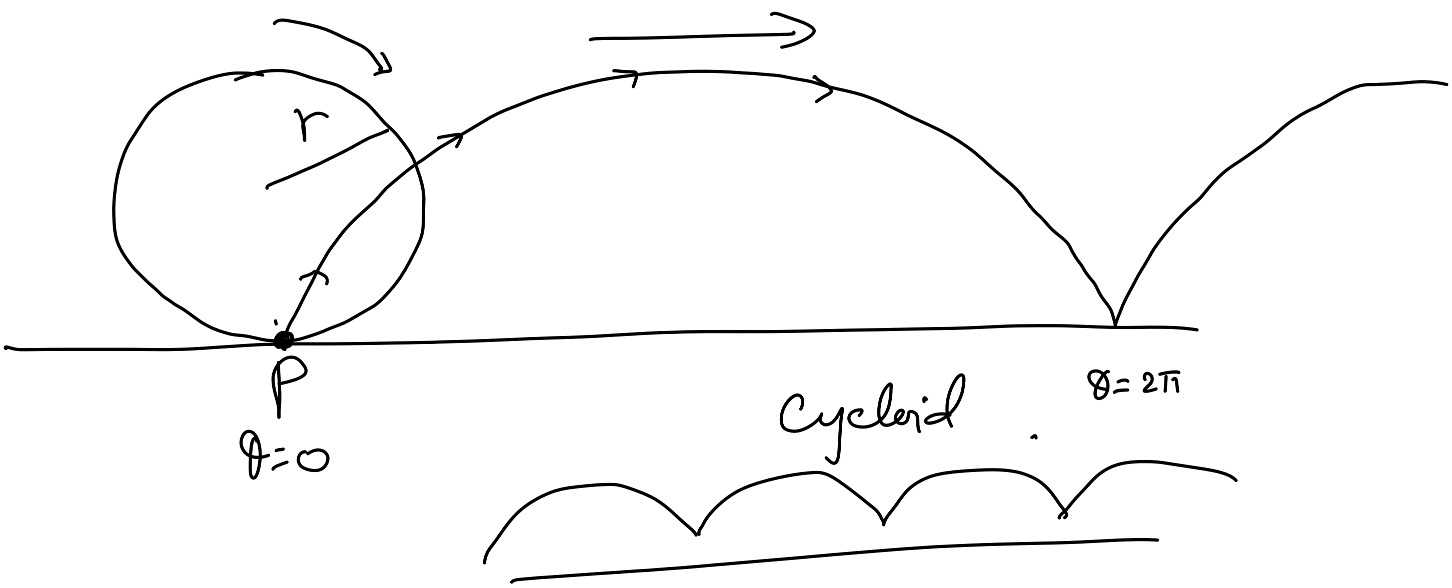
$x = 75\sqrt{3}t \Big|_{t=4.68} = 75\sqrt{3}(4.68) = 608$ ft.

Ex: The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid. If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find the parametric equations for the cycloid.

Sol:



$$\begin{cases} x = r(\theta - \sin\theta) = f(\theta) \\ y = r(1 - \cos\theta) = g(\theta) \end{cases}$$



Section 9.2 Calculus with Parametric Curves:

Given a parametric equation: $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ *for* $t \in I$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

$$y = \ldots \quad y' = \frac{dy/dt}{dx/dt}.$$

1. First derivative:

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

Ex: $\begin{cases} x = 2t^2+5 \\ y = t^3+4t-1 \end{cases} \Rightarrow y' = \frac{dy/dt}{dx/dt} = \frac{3t^2+4}{4t} = \frac{3}{4}t + t^{-1} =$

2. Second derivative:

$$y'' = \frac{d}{dx}(y') = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}}.$$

Ex: for $\begin{cases} x = 2t^2+5 \\ y = t^3+4t-1 \end{cases} \Rightarrow y'' = \,? = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3}{4}t + t^{-1}\right)}{4t} = \frac{\frac{3}{4} - t^{-2}}{4t}.$

$m =$ und.

Ex: Find point(s) where tangent lines to $\begin{cases} x = t^2+t-6 \\ y = t^2-3t-4 \end{cases}$ is either vertical or horizontal

$m = 0$.

$(x, y) = ?$

$$m = y' = \frac{dy/dt}{dx/dt} = \frac{2t-3}{2t+1} = 0$$

Vertical $\Rightarrow y' =$ undefined $\Rightarrow 2t+1 = 0 \Rightarrow t = -\frac{1}{2}$

pt: $\begin{cases} x = t^2+t-6 \\ y = t^2-3t-4 \end{cases} \Big|_{t=-\frac{1}{2}} \Rightarrow \begin{cases} x = \frac{1}{4} - \frac{1}{2} - 6 = \frac{1-2-24}{4} = -\frac{25}{4} \\ y = \frac{1}{4} + \frac{3}{2} - 4 = \frac{1+6-16}{4} = -\frac{9}{4} \end{cases} : \left(-\frac{25}{4}, -\frac{9}{4}\right)$

Horizontal $\Rightarrow y' = 0 \Rightarrow 2t-3 = 0 \Rightarrow t = \frac{3}{2}$.

pts: $\begin{cases} x = t^2+t-6 \\ y = t^2-3t-4 \end{cases} \Big|_{t=\frac{3}{2}}$ $\quad x = \frac{9}{4} + \frac{3}{2} - 6 = \frac{9+6-24}{4} = -\frac{9}{4}$, $\quad y = \frac{9}{4} - \frac{9}{2} - 4 = \frac{9-18-16}{4} = -\frac{25}{4}$

$$\left(-\frac{9}{4}, -\frac{25}{4}\right).$$

Ex: A curve C is defined by the parametric equations $x = t^2$; $y = t^3 - 3t$

$$\begin{cases} x = f(t) = t^2 \\ y = g(t) = t^3 - 3t \end{cases}$$

a) Show that C has two tangents as the point (3,0) and find their equations.

$$\text{At pt } (3,0) \Rightarrow \begin{cases} x = 3 \Rightarrow t^2 = 3 \Rightarrow t = \pm\sqrt{3} \\ y = 0 \Rightarrow t^3 - 3t = 0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow t = 0, \pm\sqrt{3} \end{cases} \quad t = \pm\sqrt{3}$$

$$\boxed{m = y' = \frac{dy/dt}{dx/dt}} = \frac{3t^2 - 3}{2t}\Bigg|_{t = \pm\sqrt{3}} = \frac{3(\pm\sqrt{3})^2 - 3}{2(\pm\sqrt{3})} = \frac{9-3}{\pm 2\sqrt{3}} = \frac{6}{\pm 2\sqrt{3}}$$

$$m = \pm\frac{3}{\sqrt{3}} = \pm\sqrt{3}\ ; \quad y - y_1 = m(x - x_1)$$

$$y - 0 = \pm\sqrt{3}(x - 3)$$

Ans: $\boxed{y = \pm\sqrt{3}(x-3)}$

b) Find the points on C where the tangent is horizontal or vertical.

(x, y)

$$m = y' = \frac{3t^2 - 3}{2t}$$

$\to$ Horizontal $\Rightarrow m = 0 \Rightarrow 3t^2 - 3 = 0 \Rightarrow t^2 - 1 = 0$, $t^2 = 1 \Rightarrow t = \pm 1$

$$\text{pts: } \begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}\Bigg|_{t = \pm 1} = \begin{cases} x = (\pm 1)^2 = 1 \\ y = (\pm 1)^3 - 3(\pm 1) \end{cases} \quad \begin{cases} 1 - 3 = -2 \\ -1 + 3 = 2 \end{cases}$$

$(1, -2)$ & $(1, 2)$

Vertical $\Rightarrow m = \text{undefined} \Rightarrow 2t = 0 \Rightarrow t = 0$.

$$\text{pt: } \begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}\Bigg|_{t=0} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (0, 0)$$

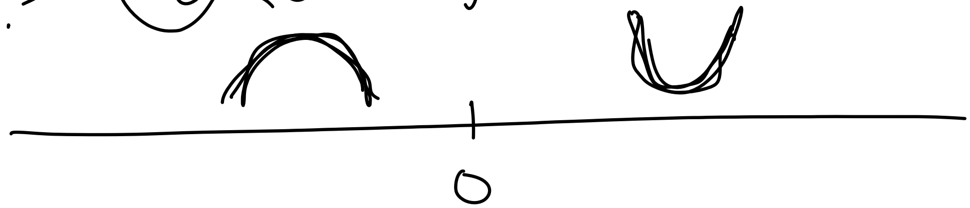
c) Determine ~~where~~ when the curve is concave upward or downward.

$$\text{Concavity} \Rightarrow y'' = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2 - 3}{2t}\right)}{2t} = \frac{\frac{d}{dt}\left(\frac{3}{2}t - \frac{3}{2}t^{-1}\right)}{2t} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t} > 0$$

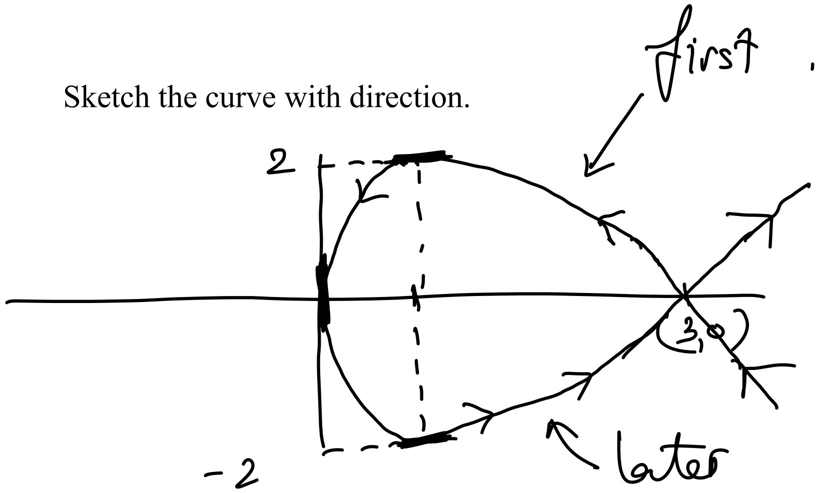
$$\left.\begin{aligned} y'' > 0 &\Rightarrow \cup \\ y'' < 0 &\Rightarrow \cap \end{aligned}\right\}$$

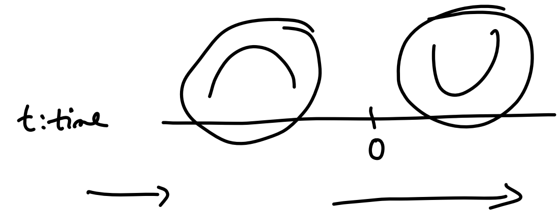
$t > 0 \Rightarrow y'' > 0 \Rightarrow \cup$.

$t < 0 \Rightarrow y'' < 0 \Rightarrow \cap$.



d) Sketch the curve with direction.





Ex: a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$; $y = r(1 - \cos\theta)$ at the point $\theta = \pi/3$

$m =$, (x_1, y_1) $y - y_1 = m(x - x_1)$

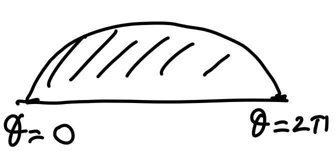
b) At what points is the tangent horizontal? When is it vertical?

Areas: of $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ for $a \le t \le b$.

$$\text{Area} = \int y\,dx = \int y \cdot \frac{dx}{dt} \cdot dt$$

$$= \int_a^b g(t) \cdot f'(t)\,dt$$

Ex: Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$



$$\text{Area} = \int y\,dx = \int y \cdot \frac{dx}{d\theta} \cdot d\theta$$

$$= \int_0^{2\pi} \left[r(1-\cos\theta) \cdot r(1-\cos\theta) \right] d\theta$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\right] d\theta$$

$$= r^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= r^2 \cdot \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta) \right]_0^{2\pi}$$

$$= r^2 \left[\frac{3}{\cancel{2}} \cdot \cancel{2}\pi \right] = \boxed{3\pi r^2}$$

Arc Length: $L = \int ds$ where $ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y=f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x=f(y) \end{cases}$ $\quad \begin{cases} x=f(t) \\ y=g(t) \end{cases}$

Consider $L = \int \sqrt{1+(y')^2}\,dx = \int \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,dx = \int \sqrt{1+\frac{(dy/dt)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot dx$

$\Rightarrow L = \int \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\cdot \frac{dx}{dt}\cdot dt = \int \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{\cancel{\frac{dx}{dt}}}\cdot \cancel{\frac{dx}{dt}}\cdot dt$

$\Rightarrow \boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt}$

Ex: Find the length of one arch of the cycloid $\begin{cases} x = r(t-\sin t) \\ y = r(1-\cos t) \end{cases}$,

Note: $\boxed{L = \int ds \Rightarrow ds = \begin{cases} \sqrt{1+(y')^2}\,dx & \text{if } y=f(x) \\ \sqrt{1+(x')^2}\,dy & \text{if } x=f(y) \\ \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt & \text{if } \begin{cases} x=f(t) \\ y=g(t) \end{cases} \end{cases}}$

Ex. Find the length of the following curve $\begin{cases} x = \cos t + \ln\left(\tan \frac{1}{2} t\right) = f(t) \\ y = \sin t = g(t) \end{cases}$; for $\pi/4 \le t \le \cancel{3\pi/4}\ \frac{\pi}{2}$.

$$L = \int ds = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$x = \cos t + \ln\left(\tan\left(\tfrac{1}{2}t\right)\right) \Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{1}{2}t\right)} \cdot \sec^2\left(\tfrac{1}{2}t\right) \cdot \frac{1}{2}.$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{2} \cdot \frac{\cancel{\cos\left(\frac{1}{2}t\right)}}{\sin\left(\frac{1}{2}t\right)} \cdot \frac{1}{\cos^2\left(\frac{1}{2}t\right)} = \frac{1}{\underbrace{2\sin\left(\frac{1}{2}t\right)\cos\left(\frac{1}{2}t\right)}_{=\sin t}} - \sin t.$$

$$\frac{dx}{dt} = \frac{1}{\sin t} - \sin t = \frac{1 - \sin^2 t}{\sin t} = \frac{\cos^2 t}{\sin t} = \cot t \cdot \cos t.$$

$$\left(\frac{dx}{dt}\right)^2 = \left(\cot t \cdot \cos t\right)^2 = \cot^2 t \cdot \cos^2 t.$$

$$+ \left(\frac{dy}{dt}\right)^2 = \left(\cos t\right)^2 = \cos^2 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cot^2 t \cos^2 t + \cos^2 t = \cos^2 t\left(\cot^2 t + 1\right) = \cos^2 t \cdot \csc^2 t.$$

$$\Rightarrow L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{\frac{\pi}{4}}^{\pi/2} \sqrt{\cos^2 t \cdot \csc^2 t}\, dt$$

$$= \int_{\pi/4}^{\pi/2} \cos t \cdot \csc t\, dt = \int \cot t\, dt = -\ln|\sin t|\Big|_{\pi/4}^{\pi/2}$$

$$= -\left(\ln\left|\sin \tfrac{\pi}{4}\right| - \ln\left|\underbrace{\sin \tfrac{\pi}{2}}_{=1}\right|\right).$$

$$= \boxed{-\ln\left(\tfrac{\sqrt{2}}{2}\right)}$$

Surface Area:

$$S = 2\pi \int r\, ds \quad \text{where } ds = \begin{cases} \sqrt{1 + (y')^2}\, dx \\ \sqrt{1 + (x')^2}\, dy \\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{cases}$$

$$r = \begin{cases} x & \text{if rotated about the } y\text{-axis} \\ y & \text{———— } x\text{-axis} \end{cases}$$

In particular for Parametric Eq$^{\underline{n}}$.

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad a \le t \le b \Rightarrow S = \begin{cases} 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \ : \ \text{Rotated about } y\text{-axis} \\ 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \ : \ \text{Rotate about the } x\text{-axis.} \end{cases}$$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex: Find the surface area of the following which rotated about the indicated axis.

a) $\begin{cases} x = e^t - t = f(t) \\ y = 4e^{t/2} = g(t) \end{cases}$, $0 \le t \le 1$ rotated about the x – axis. $\rightsquigarrow r = y$

$$S = 2\pi \int_0^1 r\,ds \text{ where } \begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \text{ match.} \\ r = y = 4e^{t/2} \end{cases}$$

$a^2 - 2ab + b^2$

$$x = e^t - t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \left(e^t - 1\right)^2 = e^{2t} - 2e^t + 1$$

$+$

$$y = 4e^{t/2} \Rightarrow \left(\frac{dy}{dt}\right)^2 = \left(4e^{t/2} \cdot \frac{1}{2}\right)^2 = \left(2e^{t/2}\right)^2 = 4e^t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1$$
$$= (e^t + 1)^2.$$

$$S = 2\pi \int_0^1 4e^{\frac{t}{2}} \sqrt{(e^t+1)^2}\,dt = 8\pi \int_0^1 e^{\frac{t}{2}} (e^t + 1)\,dt.$$

$$= 8\pi \int_0^1 \left(e^{\frac{3}{2}t} + e^{\frac{1}{2}t}\right) dt.$$

$$= 8\pi \left[\frac{2}{3} e^{\frac{3}{2}t} + 2e^{\frac{1}{2}t}\right]_0^1$$

$$= 8\pi \left[\frac{2}{3} e^{2/3} + 2e^{\frac{1}{2}} - \left(\frac{2}{3} + 2\right)\right] = \cdots = \boxed{A}$$

$=f$ $\Rightarrow r = y$

(b) $\begin{cases} x = \ln(\sec t + \tan t) - \sin t \\ y = \cos t \end{cases}$; $0 \le t \le \frac{\pi}{3}$; *about the x − axis*

$$S = 2\pi \int r\,ds \quad \text{where} \quad \begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ r = y = \cos t \end{cases}$$

match.

$$x = \ln(\sec t + \tan t) - \sin t \Rightarrow \frac{dx}{dt} = \frac{1}{\sec t + \tan t}\left(\sec t \tan t + \sec^2 t\right) - \cos t.$$

$$\frac{dx}{dt} = \frac{\sec t\,\cancel{(\tan t + \sec t)}}{\cancel{\sec t + \tan t}} - \cos t = \sec t - \cos t.$$

$$= \frac{1}{\cos t} - \cos t = \frac{1-\cos^2 t}{\cos t} = \frac{\sin^2 t}{\cos t} = \tan t \cdot \sin t.$$

$$\left(\frac{dx}{dt}\right)^2 = (\tan t \cdot \sin t)^2 = \tan^2 t \cdot \sin^2 t.$$

$$+\left(\frac{dy}{dt}\right)^2 = (-\sin t)^2 = \sin^2 t.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tan^2 t \sin^2 t + \sin^2 t.$$

$$= \sin^2 t(\tan^2 t + 1)$$

$$= \sin^2 t \cdot \sec^2 t = \sin^2 t \cdot \frac{1}{\cos^2 t} = \tan^2 t.$$

$$S = 2\pi \int_0^{\frac{\pi}{3}} \cos t \cdot \sqrt{\tan^2 t}\,dt = 2\pi \int_0^{\pi/3} \underbrace{\cos t \cdot \tan t}_{\cancel{\cos t} \cdot \frac{\sin t}{\cancel{\cos t}}}\,dt$$

$$= 2\pi \int_0^{\pi/3} \sin t\,dt = -2\pi \cos t \Big|_0^{\pi/3}$$

$$= -2\pi\left[\cos\frac{\pi}{3} - \cos(0)\right]$$

$$= -2\pi\left[\frac{1}{2} - 1\right] = (-2\pi)\left(-\frac{1}{2}\right) = \boxed{\pi}$$