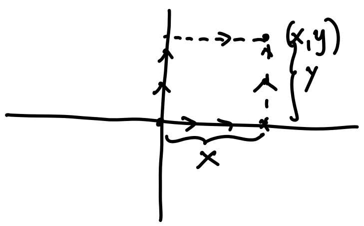
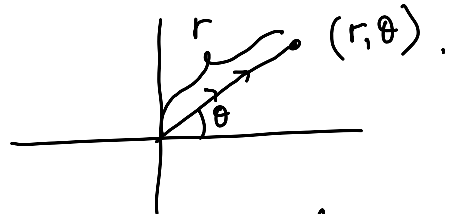
Section ~~10.2~~ 9.3 Polar Coordinates

Def: A point P is represented by the order pair (r,θ) where r is the distance from the point to the origin, and theta is the angle from the x-axis to the line connecting the point and the origin.



$\Longleftrightarrow$



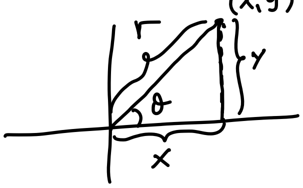
Rectangular Coord.

Polar Coord.

So for any point $(x,y) \Rightarrow (r,\theta) \Rightarrow \boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}$

Identities: $\boxed{x^2 + y^2 = r^2;\ \tan\theta = \frac{y}{x}}$

$(x,y) \approx (r,\theta)$

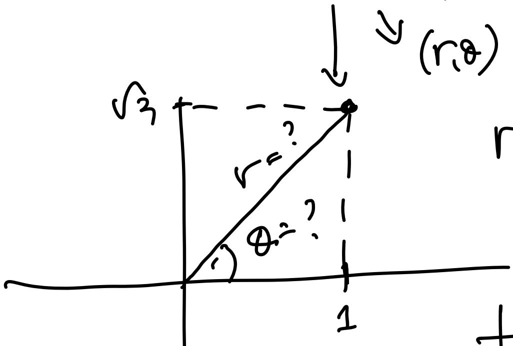


Ex: Represent the following point in Cartesian coordinate to the polar coordinates.

Rectangular

a) $(1,\sqrt{3}) = (x,y)$ b) $(1,-1)$

(r,θ)



$$r^2 = x^2 + y^2 = 1^2 + (\sqrt{3})^2 = 1 + 3 = 4$$

$$r = \sqrt{4} = 2$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

$$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

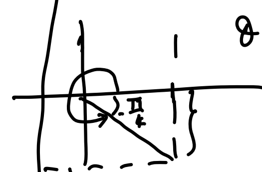
$$(1,\sqrt{3}) \approx \left(2, \frac{\pi}{3}\right)$$
$$(x,y) \qquad (r,\theta)$$

$(1,-1)$

$r^2 = 1^2 + (-1)^2 = 2.$

$r = \sqrt{2}.$

$\theta = -\frac{\pi}{4} = \frac{7\pi}{4}$

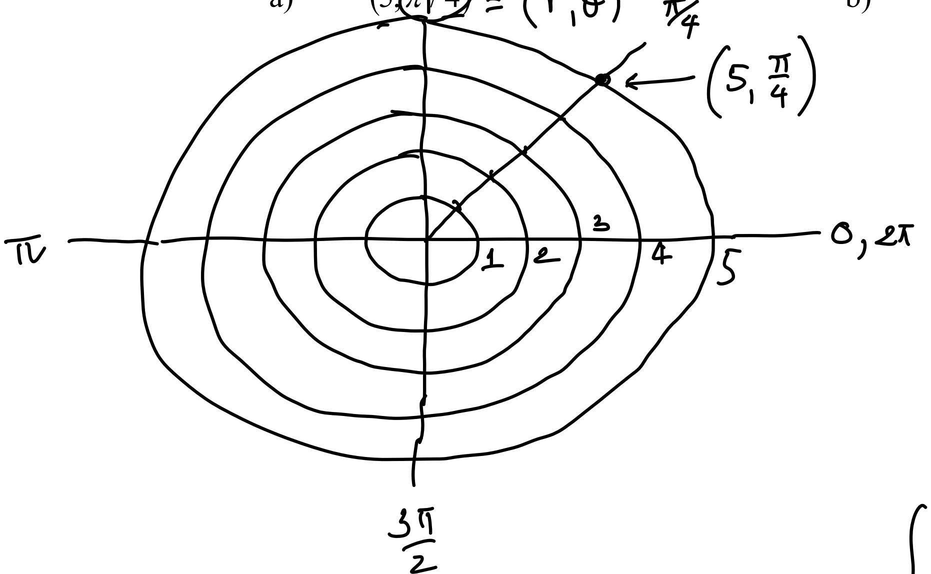


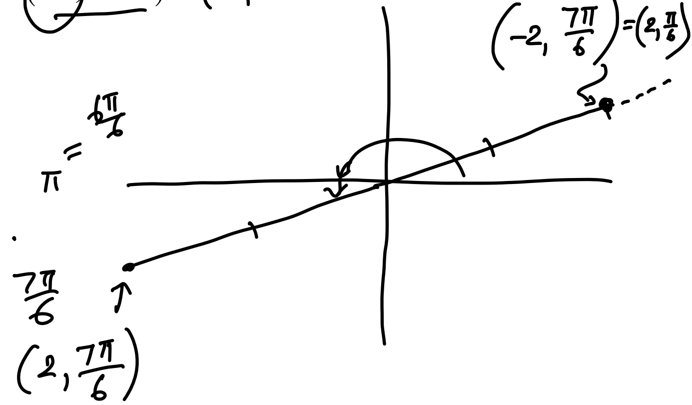
$(1,-1) \approx \left(\sqrt{2}, \frac{7\pi}{4}\right)$

$(x,y) \approx (r,\theta)$

Ex: Locate the following points in polar coordinates.

a) $(5, \pi/4) = (r, \theta)$ b) $(-2, 7\pi/6) = (r, \theta = \frac{\pi}{6})$





$$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \end{array} \right\}$$

Ex: Convert the following into rectangular coordinates:

a) $r^2 = 3r\sin\theta - 4\cos\theta$

multiply by r

$$r^3 = 3r^2\sin\theta - 4r\cos\theta.$$

$$(r^2)^{3/2} = 3r\sin\theta \cdot \sqrt{r^2} - 4r\cos\theta.$$

$$\boxed{(x^2+y^2)^{3/2} = 3y\sqrt{x^2+y^2} - 4x}$$

b) $r^3 = 2r\cos\theta - 5r\sin\theta$

$$(r^2)^{3/2} = 2r\cos\theta - 5r\sin\theta.$$

$$(x^2+y^2)^{3/2} = 2x - 5y$$

c) $$\frac{r}{1} = \frac{7}{5\cos\theta - 3\sin\theta}$$

$$r(5\cos\theta - 3\sin\theta) = 7$$

$$5r\cos\theta - 3r\sin\theta = 7$$

$$5x - 3y = 7$$

Ex: Convert to polar coordinate: r, θ

a) $x^2 + y^2 = 25$

$(r\cos\theta)^2 + (r\sin\theta)^2 = 25$

or

$x^2 + y^2 = r^2$

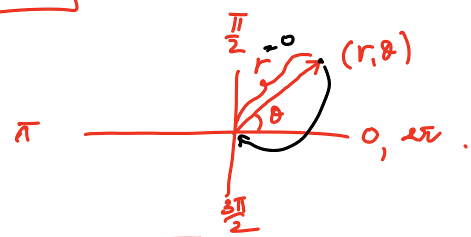
$r^2 = 25.$

$\boxed{r = 5}$ ✓

b) $7x - 5y^2 = 4$

$7r\cos\theta - 5(r\sin\theta)^2 = 4$

$7r\cos\theta - 5r^2\sin^2\theta = 4$ ✓



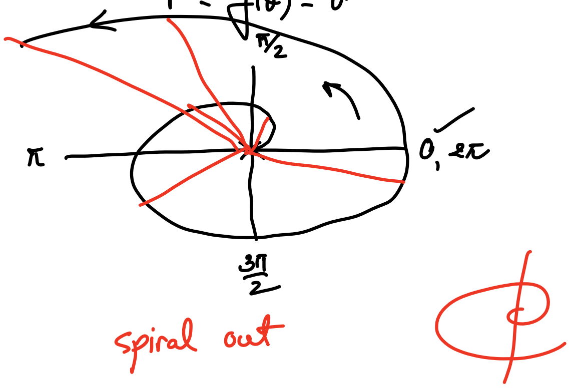
Polar Curves:

The graph of a polar equation $\boxed{r = f(\theta)}$ or more generally, $F(r, \theta) = 0$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

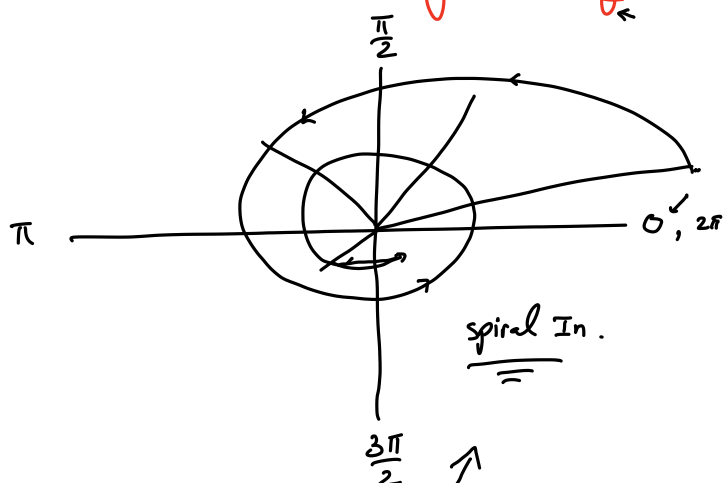
Ex: Sketch the graph of the following:

a) $r = \theta$

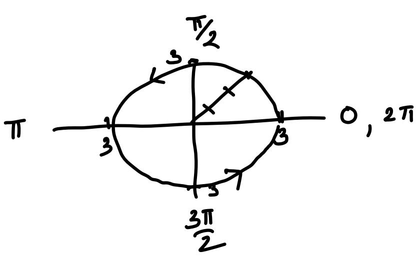
$r = f(\theta) = \theta$



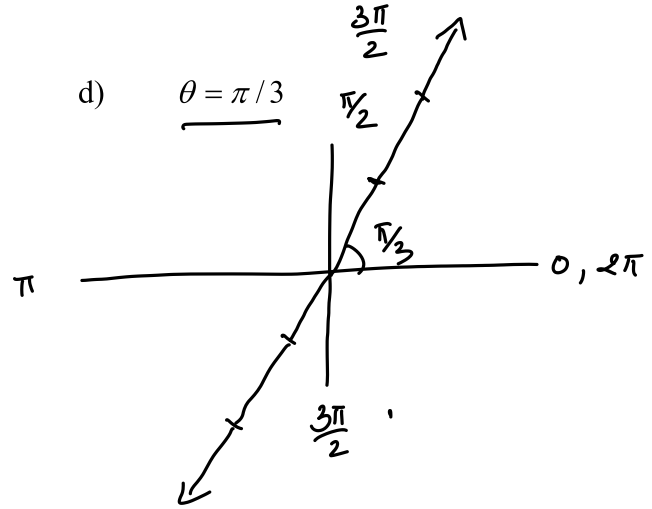
b) $r = 1/\theta$ $\Rightarrow r = f(\theta) = \frac{1}{\theta}$



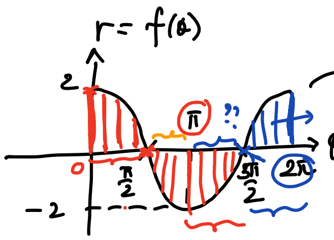
c) $r = 3$

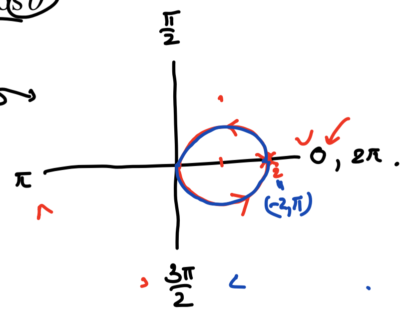


d) $\theta = \pi / 3$

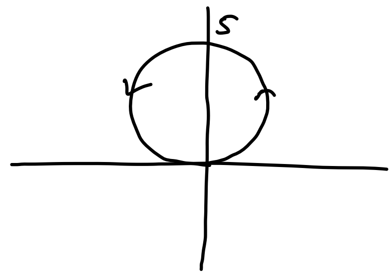


e) $r = 2\cos\theta$





f) $r = 5\sin(\theta)$



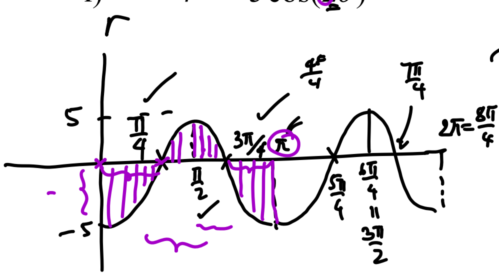
rectangular coord.

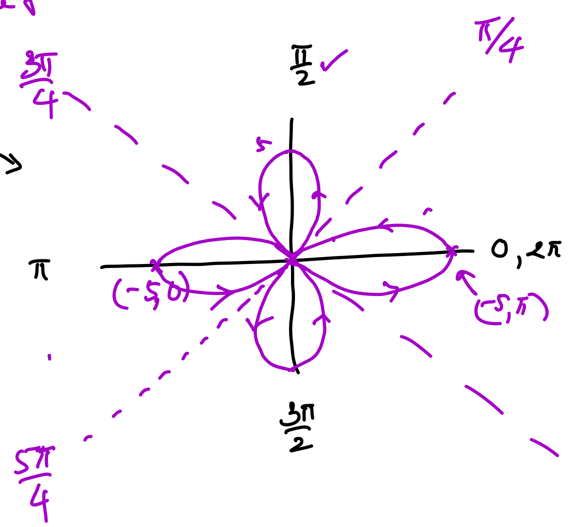
Note: cosine occurs along the horizontal line

sine ________ vertical line.

even: rose of twice the multiple of θ.

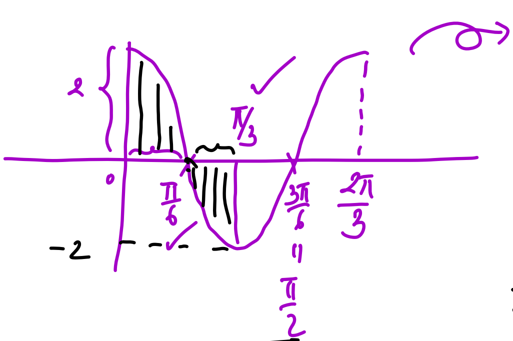
l) $r = -5\cos(2\theta)$

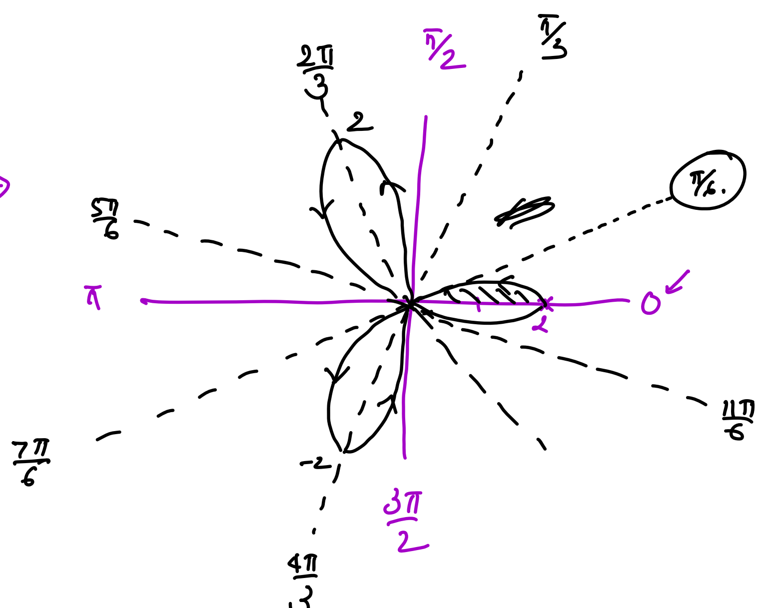




rectangular

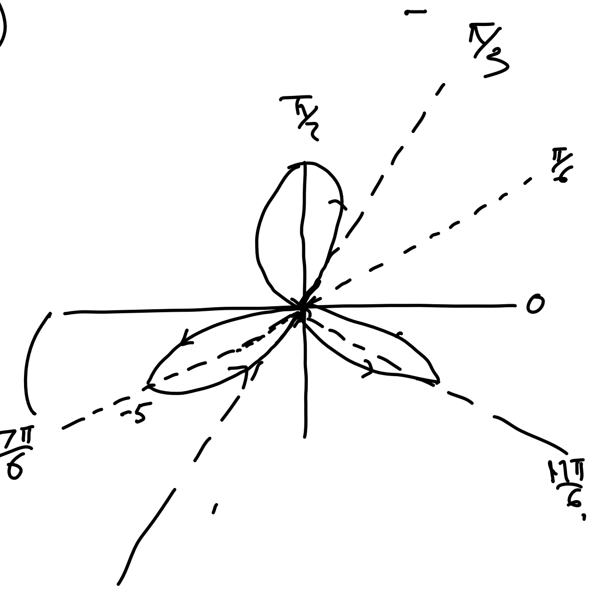
m) $r = 2\cos(3\theta)$





3 leave.

$$r = -5\sin(3\theta)$$



n) $\boxed{r = 3 - 2\cos\theta}$

$\theta = 0 \Rightarrow r = -5\sin(0) = 0$

$\theta = \frac{\pi}{6} \Rightarrow r = -5\sin\left(3\cdot\frac{\pi}{6}\right) = -5$

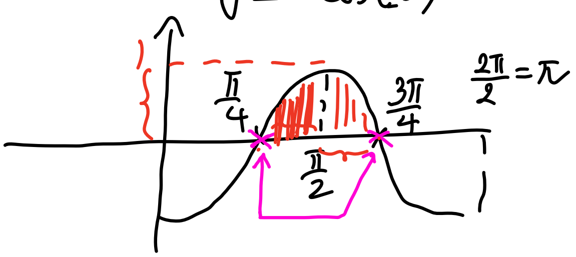
$\frac{\pi}{2}$ $\frac{\pi}{2}$

$\theta = \frac{\pi}{3} \Rightarrow r = -5\sin(\pi) = 0$ π

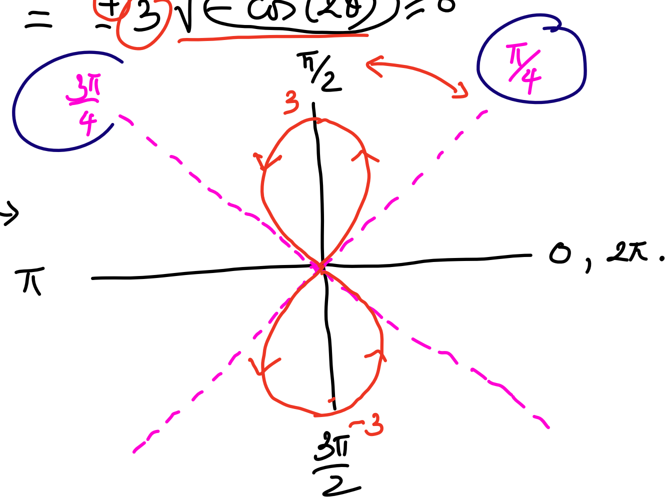
$\theta = \frac{\pi}{2} \Rightarrow r = -5\sin\left(\frac{3\pi}{2}\right) = 5$

o) $r^2 = -9\cos(2\theta) \Rightarrow r = \pm\sqrt{-9\cos(2\theta)} = \pm 3\sqrt{-\cos(2\theta)} \geq 0$

$y = -\cos(2\theta)$

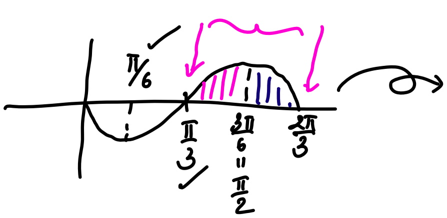


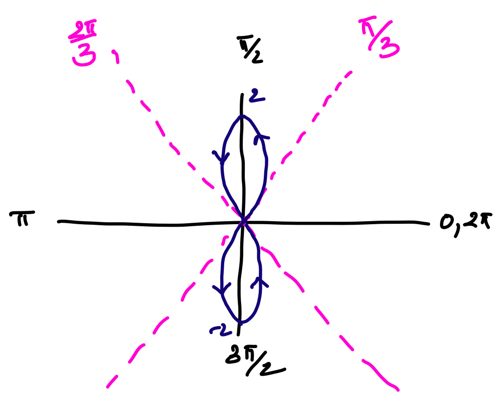
rectangular cord



p) $r^2 = -4\sin(3\theta) \Rightarrow r = \pm\sqrt{-4\sin(3\theta)} = \pm 2\sqrt{-\sin(3\theta)} \geq 0$

$y = -\sin(3\theta)$



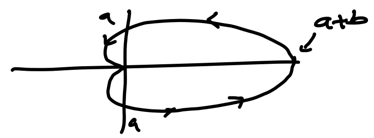


Cardioid

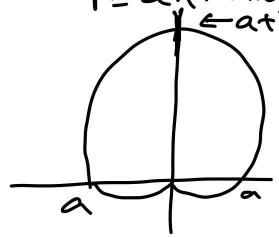
How to sketch $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$

Case 1: $a = b$ ← Perfect heart.

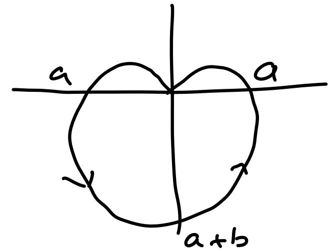
$r = a + b\cos\theta$.



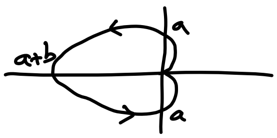
$r = a + b\sin\theta$



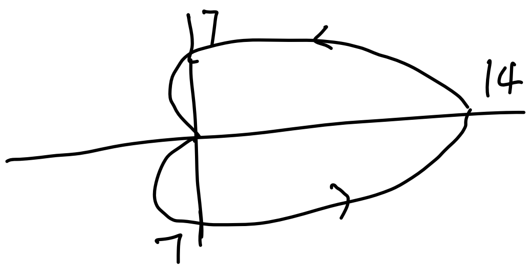
$r = a - b\sin\theta$



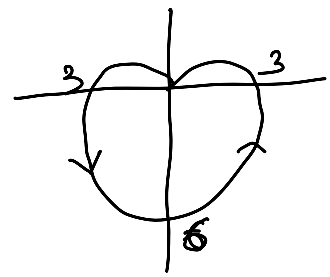
$r = a - b\cos\theta$



$r = 7 + 7\cos\theta$.

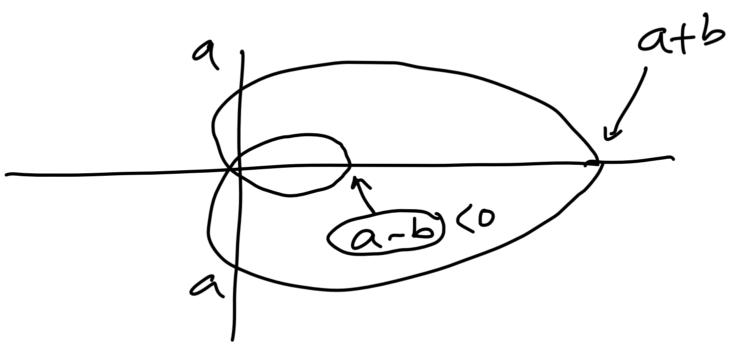


$r = 3 - 3\sin\theta$

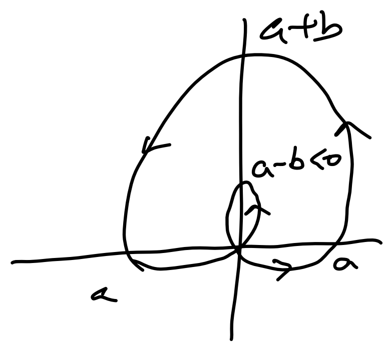


Case 2: $a < b \Rightarrow$ heart with inner loop.

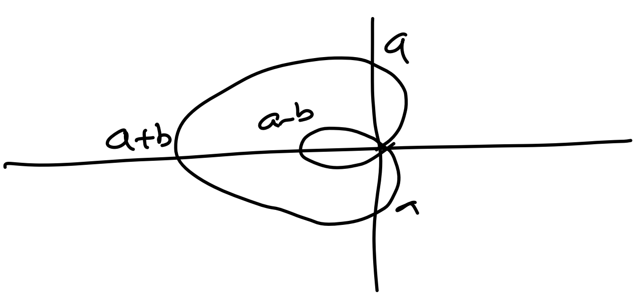
$r = a + b\cos\theta$



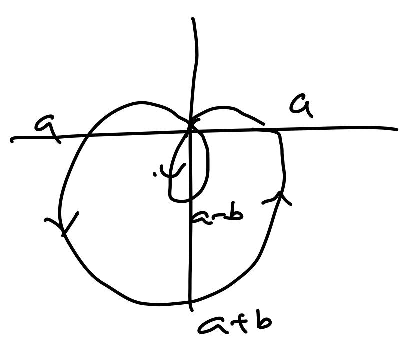
$r = a + b\sin\theta$



$r = a - b\cos\theta$



$r = a - b\sin\theta$



Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \text{ and } y = r\sin\theta = f(\theta)\sin\theta$$

$m = y' = ?$ for $r = f(\theta)$ in polar.

$x = r\cos\theta = f(\theta)\cos\theta =$

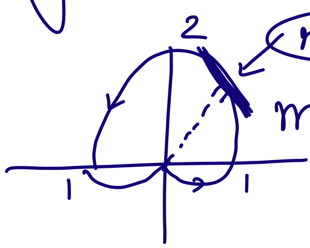
$y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Ex: a) For the cardioid $r = 1+\sin\theta$, find the slope of the tangent line where $\theta = \pi/3$

$$\boxed{y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}}$$

$r = f(\theta) = 1+\sin\theta$ $\quad m = ?$ $\quad \theta = \frac{\pi}{3}$



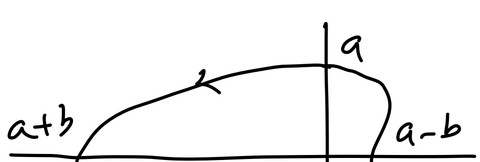
$m = -1$

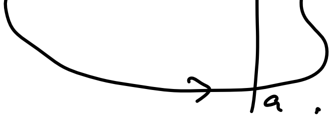
$$m = y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\cos\theta\cdot\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta}$$

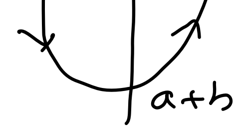
$$m = \frac{\cos\theta\,[\sin\theta + 1 + \sin\theta]}{\cos^2\theta - \sin\theta - \sin^2\theta} = \frac{\cos\theta\,[2\sin\theta + 1]}{\cos(2\theta) - \sin\theta}$$

$$m\Big|_{\theta=\frac{\pi}{3}} = \frac{\cos\frac{\pi}{3}\left[2\sin\frac{\pi}{3}+1\right]}{\cos\left(\frac{2\pi}{3}\right) - \sin\frac{\pi}{3}} = \frac{\frac{1}{2}\left[2\cdot\frac{\sqrt{3}}{2}+1\right]}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = -\frac{(\sqrt{3}+1)}{1+\sqrt{3}} = \boxed{-1}$$

$$\boxed{m = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}}$$ Given $r = f(\theta)$ in polar coord.

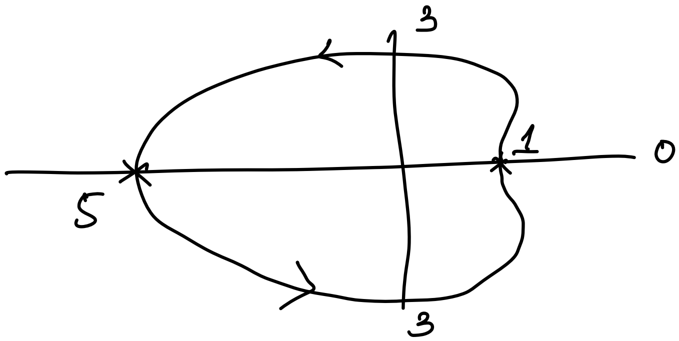


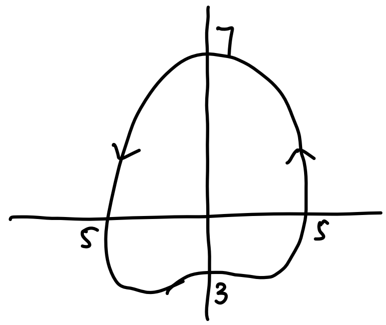




b) Find the points on the cardioid $r = 4 - 4\cos\theta$ where the tangent line is horizontal or vertical.

$$\overset{a}{r} = 3 - \overset{b}{2}\cos\theta \qquad a > b. \qquad r = 5 + 2\sin\theta$$





a)

Ex: Find $\boxed{\text{eq}^n \text{ of tangent line}}$ to $\begin{cases} x = 2t^3 - 5t + 1 = f(t) \\ y = 3\sqrt{t+3} - t = g(t) \end{cases}$ at $t = 1$.

sol: Need slope $m = ?$
point $(x_1, y_1) = (?, ?)$

$$m = y' = \frac{dy/dt}{dx/dt} = \left. \frac{\frac{3}{2}(t+3)^{-1/2} - 1}{6t^2 - 5} \right|_{t=1}$$

$$m = \frac{\frac{3}{2}(4)^{-1/2} - 1}{6 - 5} = \frac{3}{4} - 1 = -\frac{1}{4}.$$

point: $\left. \begin{cases} x_1 = 2t^3 - 5t + 1 \\ y_1 = 3\sqrt{t+3} - t \end{cases} \right|_{t=1} \Rightarrow \begin{array}{l} x_1 = 2 - 5 + 1 = -2 \\ y_1 = 3 \cdot 2 - 1 = 5 \end{array} \qquad (x_1, y_1) = (-2, 5)$

sol:
$$y - y_1 = m(x - x_1)$$
$$\boxed{y - 5 = -\frac{1}{4}(x+2)}$$

Find point(s) where the tangent line is either horizontal / vertical.

$$\begin{cases} x = \frac{2}{3}t^3 - \frac{5}{2}t^2 + 2t - 1 & = f(t) \\ y = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 20t + 4 & = g(t) \end{cases}$$

sol. $\Rightarrow$ slope $m = y' = \dfrac{dy/dt}{dx/dt}$

$$m = \frac{t^2 - t - 20}{2t^2 - 5t + 2} = 0$$

top $= 0$

$\boxed{\text{Horizontal}} \Rightarrow m = 0 \Rightarrow \underbrace{t^2 - t - 20}_{\text{top}} = 0$

$(t-5)(t+4) = 0$

$t = 5, -4.$

pts:
$$\begin{cases} x = \frac{2}{3}t^3 - \frac{5}{2}t^2 + 2t - 1 \\ y = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 20t + 4 \end{cases}$$

$\boxed{t = 5} \Rightarrow \begin{cases} x = \frac{2}{3}(5)^3 - \frac{5}{2}(5)^2 + 10 - 1 = \# \\ y = \frac{1}{3}(5)^3 - \frac{1}{2}(5)^2 - 20(5) + 4 = \# \end{cases}$

$= 3(3)$

$t = -4$

Vertical : $m =$ undefined.

$\Rightarrow 2t^2 - 5t + 4 = 0.$

$(2t \quad 4)(t - \quad 1).$